Introduction to determinantal point processes from a quantum probability viewpoint

Alex D. Gottlieb

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Abstract

Determinantal point processes on a measure space $(\mathcal{X}, \Sigma, \mu)$ whose kernels represent trace class Hermitian operators on $L^2(\mathcal{X})$ are associated to "quasifree" density operators on the Fock space over $L^2(\mathcal{X})$.

1 Introduction

This contribution has been informed and inspired by several surveys of the topic of determinantal point processes that have appeared in recent years [1, 2, 3]. The first of these, Soshnikov (2000), is inspired by the determinantal point processes that arise in random matrix theory: the set of eigenvalues of a random matrix is a realization of a determinantal point process, if the random matrix is sampled from any of the unitary-invariant ensembles of Hermitian matrices (e.g., GUE), or from uniform measure on the classical (orthogonal, unitary, or symplectic) matrix groups, or from the Ginibre Ensemble. The review by Lyons (2003) is inspired by the Transfer Current Theorem [4], which implies that the edges occurring in a randomly (uniformly) sampled spanning tree of a given finite graph G are a determinantal random subset of the edge set of G. Lyons's review concentrates on random subsets of countable sets, while Soshnikov's review is oriented to treat discrete subsets of a continuum. A very recent survey of determinantal processes (Hough *et al.* (2005)) includes the following newly-found example: the zero set of a power series with i.i.d. gaussian coefficients is a determinantal point process [5] (the radius of convergence equals 1 almost surely).

Hough *et al.* (2005) explain how a simple insight gives one a handle on number fluctuations in determinantal point processes [6, 7, 8, 9]. The insight is that, in a determinantal point process with finite expected number of points, the distribution of the number of points is equal to the distribution of the sum of independent Bernoulli(λ_j) random variables, where $0 < \lambda_j \leq 1$ are the nonzero eigenvalues of the "kernel" of the determinantal process. For example, consider the number of eigenvalues of a random $n \times n$ unitary matrix that lie in a given arc A of the unit circle. Denote this number by $\#_n A$. If the length of A is positive but less than 2π, then

$$\frac{\#_n A - \mathbb{E}\#_n A}{\sqrt{\ln n}/\pi} \tag{1}$$

is asymptotically normal with unit variance [7, 10, 11]. The subset of eigenvalues that lie in A forms a determinantal point process on A, for it is the restriction of a determinantal point process on the whole circle, hence $\#_n A$ is distributed as a sum of independent Bernoulli random variables. Thus, once one knows that the variance of $\#_n A$ is $(\ln n)/\pi^2 + o(n)$ [12, 13], the asymptotic normality of (1) follows from the Lindeberg-Feller Central Limit Theorem.

Determinantal point processes have a physical interpretation: they give the joint statistics of noninteracting fermions in a "quasifree" state. Indeed, this motivated the introduction of the concept of determinantal (or "fermion") point processes in the first place [14]. Analogously defined "boson" point processes arise in physics and are called "permanental" point processes in probabilistic writing [14, 3]. Recently, too, researchers have continued to investigate determinantal point fields from a quantum probabilistic point of view [15, 16]. We adopt this viewpoint here, and realize that the satistics of a determinantal point process with trace class Hermitian kernel $\mathcal{K}$ on $L^2(\mathcal{X})$ are those of observables on the Fock space $\mathcal{F}_0(L^2(\mathcal{X}))$ with respect to the density operator on $\mathcal{F}_0(L^2(\mathcal{X}))$ that determines the gauge-invariant quasifree state with symbol $\mathcal{K}$ on the CAR subalgebra. However, we do not dwell below on the physical interpretation, nor do we discuss states on the CAR algebra in the following. Our main objective will be to construct the determinantal point process on $\mathcal{X}$ with kernel $\mathcal{K}$, when $\mathcal{K}$ is the integral kernel of a Hermitian trace class operator on $L^2(\mathcal{X})$ with $0 \leq \|K\| \leq 1$. Once the construction is understood, the fact that the number of points in a measurable subset of $\mathcal{X}$ is distributed as a sum of independent Bernoulli random variables becomes obvious.

Finally, let us remark that determinantal/permanental processes have a couple of different interesting generalizations [17, 18]. And another rich survey of determinantal processes has just appeared in the electronic archive![19]

2 Determinantal probability measures on finite sets

Let $\mathcal{X}$ be a finite set, and let $2^{\mathcal{X}}$ denote the set of all subsets of $\mathcal{X}$. Let $\mathbb{P}$ denote a probability measure on $2^{\mathcal{X}}$, and let X be a random subset of $\mathcal{X}$ distributed as $\mathbb{P}$. Then $\mathbb{P}(X \supset E)$ denotes the measure of the class of all subsets of $\mathcal{X}$ that contain the subset E. If there exists a complex-valued function $\mathcal{K}$ on $\mathcal{X} \times \mathcal{X}$ such that

$$\mathbb{P}(X \supset \{x_1, x_2, \ldots, x_m\}) \;=\; \det \big(\mathcal{K}(x_i, x_j)\big)_{i,j=1}^{m} \tag{2}$$

for all subsets $\{x_1, x_2, \ldots, x_m\}$ of $\mathcal{X}$, where $\big(\mathcal{K}(x_i, x_j)\big)_{i,j=1}^{m}$ denotes the $m \times m$ matrix whose $(i,j)^{th}$ entry is $\mathcal{K}(x_i, x_j)$, then $\mathbb{P}$ is said to be a **determinantal probability measure** [2] with **kernel** $\mathcal{K}$. The probabilities (2) determine the probabilities $\mathbb{P}(E)$ by inclusion-exclusion, hence there can be at most one determinantal probability measure with a given kernel $\mathcal{K}$. A very basic example of a determinantal probability on $2^{\mathcal{X}}$ is the law of the random set produced by independent Bernoulli trials for the membership of each element of $\mathcal{X}$; in this case the kernel $\mathcal{K}(x', x) \;=\; \delta_{x'x}\mathbb{P}(x \in X)$.

Suppose $\mathbb{P}$ is determinantal with kernel $\mathcal{K}$. Then the complementary probability measure

$$\mathbb{P}^c(X = S) = \mathbb{P}(X = \mathcal{X} \setminus S)$$

is determinantal with kernel $\mathcal{I} - \mathcal{K}$, where $\mathcal{I}(x', x) = \delta_{x'x}$. To prove this, use the identity

$$\begin{aligned}
&\det\big(\mathcal{I}(x_i,x_j) - \mathcal{K}(x_i,x_j)\big)_{i,j=1}^{m} \\
&= 1 - \sum_{j=1}^{m} \mathcal{K}(x_j,x_j) + \sum_{1\le j_1<j_2\le m} \det\big(\mathcal{K}(x_{j_a},x_{j_b})\big)_{a,b\in\{1,2\}} \\
&\quad - \sum_{1\le j_1<j_2<j_3\le m} \det\big(\mathcal{K}(x_{j_a},x_{j_b})\big)_{a,b\in\{1,2,3\}} \\
&\quad + \cdots + (-1)^m \det\big(\mathcal{K}(x_i,x_j)\big)_{i,j=1}^{m} . \qquad (3)
\end{aligned}$$

The determinants on the right-hand side of (3) are probabilities according to (2), therefore $\det\big(\mathcal{I}(x_i,x_j) - \mathcal{K}(x_i,x_j)\big)_{i,j=1}^{m}$

$$\begin{aligned}
&= 1 - \sum_{j=1}^{m} \mathbb{P}(\{x_j\}\subset X) + \sum_{1\le j_1<j_2\le m} \mathbb{P}(\{x_{j_1},x_{j_2}\}\subset X) \\
&\quad + \cdots + (-1)^m \mathbb{P}(\{x_1,\ldots,x_m\}\subset X) \\
&= \mathbb{P}(X\subset \mathcal{X}\setminus\{x_1,\ldots,x_m\}) \qquad \text{[by inclusion-exclusion]} \\
&= \mathbb{P}((\mathcal{X}\setminus X)\supset\{x_1,\ldots,x_m\}) \\
&= \mathbb{P}^c(X\supset\{x_1,\ldots,x_m\}) . \qquad (4)
\end{aligned}$$

Suppose that $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$ is an n-member set. Define the matrix $K_{ij} = \big(\mathcal{K}(x_i,x_j)\big)_{ij=1}^{n}$. If K is a Hermitian matrix, then both K and $I - K$ must be nonnegative matrices, since all of their submatrices have nonnegative determinants by (2) and (3,4). Hence, if $\mathcal{K}$ is the kernel of a determinantal random set and K is Hermitian, then K must be the matrix of a nonnegative contraction on $\mathbb{C}^n$, i.e., necessarily $0 \le \|K\| \le 1$. Conversely, if K is the matrix of a nonnegative contraction on $\mathbb{C}^n$, then we will show that there exists a determinantal probability measure on $2^{\{1,\ldots,n\}}$ with kernel $\mathcal{K}(i,j) = K_{ij}$.

The rest of this section is devoted to the construction of a determinantal probability measure whose kernel is a nonnegative contraction. Our point of view is that there exists a density operator on the Fock space over $\mathbb{C}^n$ whose diagonal elements in the standard Fock basis give the desired probabilities.

A **density operator** is a nonnegative Hermitian operator of trace 1.

Let $\mathcal{F}(\mathbb{C}^n)$ denote the exterior algebra over $\mathbb{C}^n$, i.e.,

$$\mathcal{F}(\mathbb{C}^n) \;=\; \mathbb{C} \oplus \mathbb{C}^n \oplus \wedge^2\mathbb{C}^n \oplus \cdots \oplus \wedge^{n-1}\mathbb{C}^n \oplus \wedge^n\mathbb{C}^n \;, \tag{5}$$

where $\wedge^m\mathbb{C}^n$ denotes the m^{th} exterior power of $\mathbb{C}^n$. The exterior algebra $\mathcal{F}(\mathbb{C}^n)$ is spanned by vectors of the form $v_1 \wedge v_2 \wedge \ldots \wedge v_m$, where $v_1, \ldots, v_m$ are any m vectors in $\mathbb{C}^n$ and m is any number between 1 and n (together with an extra "vacuum vector" Ω to span the first summand). The expression $v_1 \wedge v_2 \wedge \ldots \wedge v_m$ for vectors is formally multilinear in $v_1, \ldots, v_m$ and satisfies

$$v_j \wedge \cdots \wedge v_1 \wedge \cdots \wedge v_m \;=\; -\, v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_m$$

for $j = 2, \ldots, n$. The exterior algebra $\mathcal{F}(\mathbb{C}^n)$ is 2^n dimensional and supports the inner product

$$\langle v_1 \wedge \cdots \wedge v_{m'},\; w_1 \wedge \cdots \wedge w_m \rangle \;=\; \delta_{m'm}\; \det\big(\langle v_i, w_j\rangle\big)_{ij=1}^{m}$$

(the vacuum vector is orthogonal to all $v_1 \wedge \ldots \wedge v_m$ and has unit norm). It can be shown that $\mathcal{F}(\mathbb{C}^n)$ is isomorphic to a subspace of the **Fock space**

$$\mathcal{F}_0(\mathbb{C}^n) \;=\; \mathbb{C} \oplus \mathbb{C}^n \oplus (\mathbb{C}^n \otimes \mathbb{C}^n) \oplus \cdots \oplus (\otimes^n\mathbb{C}^n)$$

via the map that assigns $1 \oplus 0_{\mathbb{C}^n} \oplus \cdots \oplus 0_{\otimes^n\mathbb{C}^n}$ to Ω and

$$0_{\mathbb{C}} \oplus \cdots \oplus 0_{\otimes^{m-1}\mathbb{C}^n} \oplus \mathcal{S}\ell[v_1, \ldots, v_m] \oplus 0_{\otimes^{m+1}\mathbb{C}^n} \oplus \cdots \oplus 0_{\otimes^n\mathbb{C}^n} \tag{6}$$

to $v_1 \wedge v_2 \wedge \ldots \wedge v_m$. In (6), $\mathcal{S}\ell[v_1, \ldots, v_m]$ denotes the Slater determinant

$$\mathcal{S}\ell[v_1, \ldots, v_n] \;=\; \frac{1}{\sqrt{n!}} \sum_{\pi \in \mathcal{S}_m} \mathrm{sgn}(\pi)\; U_\pi(v_1 \otimes v_2 \otimes \cdots \otimes v_m) \;, \tag{7}$$

where $\mathcal{S}_m$ denotes the group of permutations of $\{1, \ldots, m\}$ and U_π is the unitary operator defined on $\otimes^m\mathbb{C}^n$ when $\pi \in \mathcal{S}_m$ by the condition that

$$U_\pi(w_1 \otimes w_2 \otimes \cdots \otimes w_m) \;=\; w_{\pi^{-1}(1)} \otimes w_{\pi^{-1}(2)} \otimes \cdots \otimes w_{\pi^{-1}(m)} \tag{8}$$

for all $w_1, \ldots, w_m \in \mathbb{C}^n$. Henceforth, we identify the exterior algebra $\mathcal{F}(\mathbb{C}^n)$ with this subspace of $\mathcal{F}_0(\mathbb{C}^n)$, and call it the "fermion Fock space."

An orthonormal basis of $\mathcal{F}(\mathbb{C}^n)$, called a **Fock basis** or "occupation number" basis, can be built using any ordered orthonormal basis $\mathrm{v} = (v_1, \ldots, v_n)$ of $\mathbb{C}^n$. The vectors of the Fock basis can be conveniently indexed by subsets of $\{1, \ldots, n\}$: the empty subset of $\{1, \ldots, n\}$ corresponds to the vacuum vector Ω and a nonempty subset $\{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}$ with $j_1 < \ldots < j_m$ corresponds to the vector $v_{j_1} \wedge \cdots \wedge v_{j_m}$. That is, the orthonormal set $\{f_{\mathrm{v}}(S) \mid S \subset \{1, \ldots, n\}\}$ is a basis for $\mathcal{F}(\mathbb{C}^n)$, where $f_{\mathrm{v}}(\{\}) = \Omega$ and $f_{\mathrm{v}}(S) = v_{j_1} \wedge \cdots \wedge v_{j_m}$ when $S = \{j_1, \ldots, j_m\}$ with $j_1 < \ldots < j_m$.

Suppose K is a nonnegative contraction on $\mathbb{C}^n$ and let $\mathrm{v} = (v_1, \ldots, v_n)$ be an ordered orthonormal basis of $\mathbb{C}^n$ such that $Kv_j = \lambda_j v_j$ for all j. Let D_K denote the density operator

$$D_K \;=\; \sum_{S \subset \{1,\ldots,n\}} \Big\{ \prod_{k \in S} \lambda_k \prod_{k \notin S} (1 - \lambda_k) \Big\} \, \big\langle f_{\mathrm{v}}(S), \,\cdot\, \big\rangle f_{\mathrm{v}}(S) \tag{9}$$

on $\mathcal{F}(\mathbb{C}^n)$, where $\big\langle f_{\mathrm{v}}(S), \,\cdot\, \big\rangle f_{\mathrm{v}}(S)$ denotes the rank-one orthogonal projector onto the span of $f_{\mathrm{v}}(S)$.

Proposition 1 *Let K be a nonnegative contraction on $\mathbb{C}^n$ and let D_K denote the associated density operator (9) on the Fock space $\mathcal{F}(\mathbb{C}^n)$. Then, for all ordered orthonormal bases $\mathrm{w} = (w_1, \ldots, w_n)$ of $\mathbb{C}^n$,*

$$S \;\longmapsto\; \big\langle f_{\mathrm{w}}(S), D_K f_{\mathrm{w}}(S) \big\rangle \tag{10}$$

is a determinantal probability measure on $2^{\{1,\ldots,n\}}$ with kernel $\mathcal{K}(i,j) = \langle K w_i, w_j \rangle$.

Proof: We first define the "second quantization" maps from operators A on $\otimes^m \mathbb{C}^n$ to operators $\Gamma_m[A]$ on the Fock space $\mathcal{F}_0(\mathbb{C}^n)$, and the dual maps from density operators D on $\mathcal{F}_0(\mathbb{C}^n)$ to m-particle "correlation operators" $\mathrm{K}_m[D]$ on $\otimes^n \mathbb{C}^n$.

Let $\mathcal{J}(m,k)$ denote the set of injections of $\{1, \ldots, m\}$ into $\{1, \ldots, k\}$. The cardinality of $\mathcal{J}(m,k)$ is $k^{[m]} \equiv k(k-1)\cdots(k-m+1)$, the m^{th} factorial power of k. For any operator A on $\otimes^m \mathbb{C}^n$, and any injection $j \in \mathcal{J}(m,k)$ with $k \geq m$, we define the operator

$$A^{(j)} \;=\; U_{(1j_1)(2j_2)\cdots(mj_m)} (A \otimes I \otimes \cdots \otimes I) U_{(1j_1)(2j_2)\cdots(mj_m)}$$

on $\otimes^k \mathbb{C}^n$, where $U_{(1j_1)(2j_2)\cdots(mj_m)}$ denotes the permutation operator (8) for the product of disjoint transpositions $(1j_1)(2j_2)\cdots(mj_m)$. Define $\Gamma_m[A]$ on $\mathcal{F}_0(\mathbb{C}^n)$ by

$$\Gamma_m[A] \;=\; 0_{\mathbb{C}} \oplus \cdots \oplus 0_{\otimes^{m-1}\mathbb{C}^n} \oplus \sum_{j\in\mathcal{J}(m,m)} A^{(j)} \oplus \cdots \oplus \sum_{j\in\mathcal{J}(m,n)} A^{(j)} \;.$$

A density operator D on $\mathcal{F}(\mathbb{C}^n)$ extends to a density operator $D\oplus 0$ on $\mathcal{F}_0(\mathbb{C}^n)$, which we will denote by D as well. The map $A \longmapsto \mathrm{Tr}(D\Gamma_m[A])$ is a linear functional on the space of linear operators on $\otimes^m\mathbb{C}^n$. Therefore there exists a unique operator $\mathrm{K}_m[D]$ on $\otimes^m\mathbb{C}^n$ such that

$$\mathrm{Tr}(\Gamma_m[A]D) = \mathrm{Tr}(A\mathrm{K}_m[D])$$

for all linear operators A on $\otimes^m\mathbb{C}^n$. In physics language, $\mathrm{K}_m[D]$ is the m-particle correlation operator for the state with density operator D. If D_K is defined as in (9), the key identity

$$\mathrm{K}_m[D_K] \;=\overbrace{K\otimes K\otimes\cdots\otimes K}^{m\ \text{times}}\sum_{\pi\in\mathcal{S}_m}\mathrm{sgn}(\pi)U_\pi \tag{11}$$

may verified by comparing matrix elements of both sides with respect to the basis $\{v_{j_1}\otimes\cdots\otimes v_{j_m} |\ j_1,\ldots,j_m\in\{1,\ldots,n\}\}$.

Given an ordered orthonormal basis $\mathrm{w}=(w_1,\ldots,w_n)$, let P_j^{w} denote the projector $\langle w_j,\cdot\rangle w_j$ for $j=1,\ldots,n$. For distinct $x_1,\ldots,x_m$, the operator $\Gamma_m[P^{\mathrm{w}}_{x_1}\otimes\cdots\otimes P^{\mathrm{w}}_{x_m}]$ is diagonal in the Fock basis $\{f_{\mathrm{w}}(S)\mid S\subset\{1,\ldots,n\}\}$, and

$$\Gamma_m[P^{\mathrm{w}}_{x_1}\otimes\cdots\otimes P^{\mathrm{w}}_{x_m}]f_{\mathrm{w}}(S) \;=\; \begin{cases} f_{\mathrm{w}}(S) & \text{if } \{x_1,\ldots,x_m\}\subset S \\ \quad 0 & \text{otherwise.}\end{cases}$$

Let $\mathbb{P}(X = S)$ denote the probability (10). Then

$$\begin{aligned}
\mathbb{P}(X \supset \{x_1, x_2, \ldots, x_m\}) &= \sum_{S \supset \{x_1, x_2, \ldots, x_m\}} \mathbb{P}(X = S) \\
&= \sum_{S \supset \{x_1, x_2, \ldots, x_m\}} \big\langle f_{\mathrm{w}}(S), D_K f_{\mathrm{w}}(S) \big\rangle \\
&= \mathrm{Tr}(\Gamma_m[P^{\mathrm{w}}_{x_1} \otimes \cdots \otimes P^{\mathrm{w}}_{x_m}] D_K) \\
&= \mathrm{Tr}((P^{\mathrm{w}}_{x_1} \otimes \cdots \otimes P^{\mathrm{w}}_{x_m}) \mathrm{K}_m[D_K]) \\
&= \mathrm{Tr}\Big((P^{\mathrm{w}}_{x_1} K \otimes \cdots \otimes P^{\mathrm{w}}_{x_m} K) \sum_{\pi \in \mathcal{S}_m} \mathrm{sgn}(\pi) U_\pi\Big) \\
&= \det\big(\langle K w_{x_i}, w_{x_j} \rangle\big)_{i,j=1}^{m} \;=\; \det\big(\mathcal{K}(x_i, x_j)\big)_{i,j=1}^{m} .
\end{aligned}$$

This proves the proposition. □

3 Determinantal finite point processes

A **finite point process on** $\mathcal{X}$ is a random finite subset of a space $\mathcal{X}$. Let Σ be a σ-field of measurable subsets of $\mathcal{X}$ [20]. A finite point process on $(\mathcal{X}, \Sigma)$ is specified by the probabilities $p_0, p_1, p_2, \ldots$ that there are $0, 1, 2, \ldots$ points in the configuration, and, for each n such that $p_n \neq 0$, a symmetrical conditional probability measure ρ_n on $(\mathcal{X}^n, \otimes^n \Sigma)$ [21]. Now let μ be any positive "reference" measure on $(\mathcal{X}, \Sigma)$. A finite point process is **determinantal on** $(\mathcal{X}, \Sigma, \mu)$ with kernel $\mathcal{K} : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{C}$ if

$$\mathbb{E}\Big(\prod_{j=1}^{m} \#(X \cap E_j)\Big) \;=\; \int_{E_1} \cdots \int_{E_m} \det\big(\mathcal{K}(x_i, x_j)\big)_{ij=1}^{m} \mu(dx_1) \cdots \mu(dx_m) \qquad (12)$$

for all disjoint, measurable $E_1, \ldots, E_m$, $m \geq 1$ [3]. If $\mathcal{K}(x, y)$ is the standard version of the integral kernel of a nonnegative trace class contraction K on $L^2(\mathcal{X}, \Sigma, \mu)$, then there exists a unique [22] determinantal point process on $(\mathcal{X}, \Sigma, \mu)$ with kernel $\mathcal{K}(x, y)$. Conversely, if the kernel of a determinantal point process on $\mathcal{X}$ is the integral kernel of a trace class Hermitian operator K on $L^2(\mathcal{X})$, then K must be a nonnegative contraction [23].

In this section we construct the determinantal point process on $\mathcal{X}$ whose kernel is the standard kernel (17) of a given trace class operator K on $L^2(\mathcal{X})$ with $0 \leq \|K\| \leq 1$. This is accomplished by constructing a density operator on $\mathcal{F}(L^2(\mathcal{X}))$ as we have done in the preceding section — our quantum probabilistic point of view. There are many other ways to accomplish the same end, with or without our point of view. The original approach of Macchi (1975) was to start with a formula for the Janossy densities [24] of the desired point process, and then to verify (12) for that process. Soshnikov (2000) attacks the problem by first showing that certain Fredholm determinants involving K define factorial moment generating functions for finite families of random variables $\{\#(X \cap E_j) | E_j \in \Sigma\}$, and then constructing the desired determinantal point process via Kolmogorov extension from its finite dimensional distributions. Lyons (2003) uses the geometry of Fock space, but only in the case where K is a finite rank projector, then dilates nonnegative contractions to projections on a larger space to handle the general case. Hough *et al.* (2005) verify directly that kernels of finite rank projectors yield determinantal point processes, then treat the general case as a mixture, in the probabilistic sense, of determinantal processes with projector kernels.

Any density operator on $\mathcal{F}(L^2(\mathcal{X}, \Sigma, \mu))$ of the form $D = \oplus D_n$ defines a finite point process on $\mathcal{X}$ as follows. $p_n = \mathrm{Tr}(D_n)$ is the probability of the event the configuration has exactly n points. The measure ρ_n is absolutely continuous with respect to $\otimes^n \mu$ and it is defined by way of the isomorphism

$$\otimes^n L^2(\mathcal{X}, \Sigma, \mu) \;\cong\; L^2(\mathcal{X}^n, \otimes^n \Sigma, \otimes^n \mu).$$

Regarding D_n as an operator on $L^2(\mathcal{X}^n, \otimes^n \Sigma, \otimes^n \mu)$, define $\rho_n(E) = p_n^{-1} \mathrm{Tr}(D_n \mathcal{M}_E)$, where $\mathcal{M}_E$ denotes the operator on $L^2(\mathcal{X}^n)$ of multiplication by the indicator function of $E \in \otimes^n \Sigma$.

Given a trace class nonnegative contraction K on $L^2(\mathcal{X})$, a density operator D_K on $\mathcal{F}(L^2(\mathcal{X}))$ may be defined using the spectral information in K as was done in (9) above:

$$D_K \;=\; \sum_{n=0}^{\infty} \sum_{\substack{S \subset \{1,2,\ldots\}: \\ \#S = n}} \Big\{ \prod_{k \in S} \lambda_k \prod_{k \notin S} (1 - \lambda_k) \Big\} \, \langle f(S), \,\cdot\, \rangle f(S) \;, \tag{13}$$

where $\lambda_1 \geq \lambda_2 \geq \ldots$ are the eigenvalues of K and $\{f(S)\}$ is the Fock basis constructed from the eigenvectors of K (really, any extension of an orthonormal system of eigenvectors of K to an orthonormal basis of $L^2(\mathcal{X})$). D_K has the form $\oplus (D_K)_n$ and it can be verified [25] that the m-particle correlation operator $\mathrm{K}_m[D_K]$ exists and satisfies the key identity

(11). Let p_n and ρ_n be as defined above, for D_K. Let $\mathbb{E}$ denote the expectation with respect to random point process defined by these p_n and ρ_n. Then

$$\mathbb{E}\Big(\prod_{j=1}^{m} \#(X \cap E_j)\Big) \;=\; \mathrm{Tr}(\Gamma_m[P_1 \otimes \cdots \otimes P_m] D_K)\ , \tag{14}$$

where P_j denotes the orthogonal projector $\mathcal{M}_{E_j}$ on $L^2(\mathcal{X})$, for both sides of (14) equal $\sum\limits_{n\geq m} p_n \int_{\mathcal{X}^n} \prod_{j=1}^m \#(\{x_1,\ldots,x_n\} \cap E_j)\rho_n(dx_1 \cdots dx_n)$. But

$$\mathrm{Tr}(\Gamma_m[P_1 \otimes \cdots \otimes P_m] D_K) \;=\; \mathrm{Tr}\Big((P_1 K \otimes \cdots \otimes P_m K) \sum_{\pi \in \mathcal{S}_m} \mathrm{sgn}(\pi) U_\pi\Big)\ , \tag{15}$$

since $\mathrm{K}_m[D_K] = \big(\otimes^m K\big) \sum\limits_{\pi\in\mathcal{S}_m} \mathrm{sgn}(\pi) U_\pi$. Finally, one may verify [26] that

$$\mathrm{Tr}\Big(\big(\otimes_{j=1}^m P_j K\big) \sum_{\pi\in\mathcal{S}_m} \mathrm{sgn}(\pi) U_\pi\Big) \;=\; \int_{E_1} \cdots \int_{E_m} \det\big(\mathcal{K}(x_i,x_j)\big)_{ij=1}^{m} \mu(dx_1)\cdots\mu(dx_m) \tag{16}$$

if $\mathcal{K}(x,y)$ is the usual version of the integral kernel of K, i.e.,

$$\mathcal{K}(x,y) \;=\; \sum_j \lambda_j \phi_j(x)\overline{\phi_j(y)} \tag{17}$$

where $K\phi_j = \lambda_j \phi_j$ and $\sum \lambda_j = \mathrm{Tr} K$. Equations (14) - (16) imply (12) holds; the finite point process defined through D_K is determinantal with kernel $\mathcal{K}$.

Now that we have constructed the process, we can see immediately from (13) that the total number of points in a random configuration is distributed as the sum of Bernoulli(λ_j) random variables. In particular, $\prod(1-\lambda_k)$ is the probability that there are no points at all. This equals the Fredholm determinant $\mathrm{Det}(I-K)$. Let E be a measurable subset of $\mathcal{X}$. It is not difficult to check via (12) that the determinantal point process on $(E, \Sigma|_E, \mu|_E)$ with kernel $K_E \equiv \mathcal{M}_C K \mathcal{M}_C\big|_E$ is the restriction to E of the determinantal point process with kernel K on $\mathcal{X}$. Hence the probability that there are no points in E equals the Fredholm determinant $\mathrm{Det}(I-K_E)$. In the context of random matrix theory, this yields formulas for the spacing distributions of eigenvalues [27, 28].

In case $\|K\| < 1$, set $L = (I-K)^{-1}K$. It is easy to check from (13) that

$$(D_K)_n \;=\; \frac{1}{n!}\,\mathrm{Det}(I-K)\big\{\otimes^n L\big\}\sum_{\pi\in\mathcal{S}_n}\mathrm{sgn}(\pi)U_\pi$$

by comparing matrix elements of both sides of this identity with respect to an eigenbasis of K. This identity yields the determinantal formulas for the Janossy densities [14, 21].

4 Determinantal processes of infinitely many points

Suppose that $\mathcal{X}$ is a locally compact Hausdorff space satisfying the second axiom of countability [29], and let Σ denote the Borel field of $\mathcal{X}$. In this context, a point process is a random nonnegative integer-valued Radon measure (a Radon measure is a Borel measure which is finite on any compact set) [3]. Let μ be a σ-finite Radon measure on $\mathcal{X}$ [9]. A point process on $(\mathcal{X},\Sigma,\mu)$ is **determinantal** with kernel $\mathcal{K}$ if (12) holds.

Most work on determinantal processes with infinitely many points has been done for the cases where $\mathcal{X}$ is a countable set with the discrete topology and μ is counting measure, or $\mathcal{X}$ is a connected open subset of $\mathbb{R}^d$ and μ is Lebesgue measure, or $\mathcal{X}$ is a finite disjoint union or Cartesian product of said spaces. If K is a *locally* trace class Hermitian operator on $L^2(\mathcal{X})$ such that $0 \le \|K\| \le 1$, then (a version of) its integral kernel is the kernel of a determinantal point process on $\mathcal{X}$ [1]. This point process is the limit in distribution of the determinantal processes with kernels $\mathbf{1}_C(x)\mathcal{K}(x,y)\mathbf{1}_C(y)$, where C ranges over an increasing family of compact subsets of $\mathcal{X}$. Conversely, if the kernel of a locally trace class Hermitian operator K defines a determinantal point process, then K must be a nonnegative contraction [1, 14].

The case of a countably infinite set $\mathcal{X}$ with counting measure is treated in detail in Lyons (2003). In this pleasant special case, Hermitian operators on $L^2(\mathcal{X},2^{\mathcal{X}},\#)$ are automatically locally trace class. On the other hand, equations (2) and (3)-(4) readily imply that K and $I-K$ are both nonnegative operators. Therefore, the kernel of a Hermitian operator K on $L^2(\mathcal{X},2^{\mathcal{X}},\#)$ is the kernel of a determinantal point process on $(\mathcal{X},2^{\mathcal{X}},\#)$ if and only if K is a nonnegative contraction.

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[13] Note that the asymptotic variance of the number of eigenvalues in an arc is independent of the length of the arc! Another astounding fact is that the numbers $\#I$ and $\#J$ of eigenvalues in two intervals I and J are asymptotically uncorrelated if I and J have no endpoints in common. See Wieand (2002) and Diaconis (2003).

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[20] The space $\mathcal{X}$ is ordinarily assumed to be a nice topological space: in Daley and Vere-Jones (2003) it is a complete separable metric space and in other works [8, 18] it is locally compact and second countable. But for our purposes in this section, we only need a measure space $(\mathcal{X}, \Sigma, \mu)$.

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[22] Uniqueness follows from a sort of inclusion-exclusion formula for the measures $p_n \rho_n$ in terms of all "m-point correlation functions" $\det\big(\mathcal{K}(x_i, x_j)\big)_{ij=1}^{m}$ with $m \geq n$.

[23] The hypothesis that K is trace class is used to prove that $\|K\| \leq 1$ in general [1, 3]. However, this hypothesis is not required if $\mathcal{X}$ is a locally compact space and $\mathcal{K}$ is continuous (and μ is a Radon measure on the Borel field). In such cases, if the (continuous) kernel of a determinantal point process on $\mathcal{X}$ is the integral kernel of a bounded — but not necessarily trace class — Hermitian operator K on $L^2(\mathcal{X})$, then necessarily $0 \leq \|K\| \leq 1$.

[24] The Janossy densities are the densities of the measures $n!p_n\rho_n$.

[25] See Proposition 2.2 of archived manuscript math-ph/0303070 .

[26] Equation (16) is the same as (1.27) of Soshnikov (2000), but it can be confirmed directly by substituting (17) in (16).

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[29] Such are the spaces considered in the work of A. Lenard on correlation densities for infinitely many particles (viz. Theorem 1 in [1]).