Vector Fields and Classical Theorems of Topology

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In this talk we prove a collection of classical theorems using the concept of the index of a vector field. They are:

The Intermediate Value Theorem

The Fundamental Theorem of Algebra

Rouche's Theorem

The Gauss-Lucas Theorem

The Gauss-Bonnet Theorem

The Brouwer Fixed Point Theorem

The Borsuk-Ulam Theorem

The Jordan Curve Theorem

Gottlieb's Theorem

The Poincare-Hopf Theorem.

The reason for this exercise is to argue that the following equation, which we call the "Law of Vector Fields," is fated to play a central role in mathematics. These theorems are all easy consequences of the law of vector fields. The proofs are so mechanical that one could say that the Law of Vector Fields is a generalization of each of them.

The Law of Vector Fields is the following: Let M be a compact smooth manifold and let V be a vector field on M so that $V(m) \neq \vec{0}$ for all m on the boundary ∂M of M. Then ∂M contains an open set $\partial_- M$ which consists of all $m \in \partial M$ so that $V(m)$ points inside. We define a vector field, denoted $\partial - V$ on $\partial_- M$, so that for every $m \in \partial_- M$ we have $\partial_- V(m) =$ Projection of $V(m)$ tangent to $\partial_- M$. Under these conditions we have

$$\text{Ind } V + \text{Ind } \partial_- V = \chi(M) \tag{1}$$

where Ind V is the index of the vector field and $\chi(M)$ is the Euler characteristic of M. ([M], [G_2-G_5], [P]).

The Law of Vector Fields can be used to define the index of vector fields, so the whole of index theory follows from (1). The definition of index is not difficult, but proving it is well-defined is a little involved [G-S]. The definition proceeds as follows:

a) The index of an empty vector field is zero.

b) If M is a finite set of points and V is defined on all of the M (the vectors are necessarily zero), then $\text{Ind}(V) =$ number of points in M.

c) If V is a *proper* vector field on a compact M, by which we mean V has no zeros on ∂M, then we set

$$\text{Ind } V = \chi(M) - \text{Ind}(\partial_- V).$$

d) If V is defined on the closure of an open subset U of a smooth manifold M so that the set of zeros Z is compact and $Z \subset U$, then we say V is a *proper vector* field. The index Ind V is defined to be $\text{Ind}(V|M)$ where M is any compact manifold such that $Z \subset M \subset U$.

e) If C is a connected component of Z and C is compact and open in Z define $\text{Ind}_C(V)$ to be the index of V restricted to an open set containing C and no other zeros of V.

A key idea in proving this definition is well-defined is a generalization of the concept of homotopy which we call otopy. An *otopy* is what $\partial_- V$ undergoes when V undergoes a homotopy. The formal definition is as follows: An *otopy* is a vector field V defined on the closure of an open set $T \subset M \times I$ so that $V(m,t)$ is tangent to the slice $M \times t$. The otopy is *proper* if the set of zeros Z of V is compact and contained in T. The restriction of V to $M \times 0$ and $M \times 1$ are said to be properly otopic vector fields. Proper otopy is an equivalence relation.

The following properties hold for the index:

(2) Let M be a connected manifold. The proper otopy classes of proper vector fields on M are in one to one correspondence via the index to the integers. If M is a compact manifold with a connected boundary, then a vector field V is properly homotopic to W if and only if Ind $V =$ Ind W.

(3) $\text{Ind}(V|A \cup B) = \text{Ind}(V|A) + \text{Ind}(V|B) - \text{Ind}(V|A \cap B)$

(4) $\text{Ind}(V \times W) = \text{Ind}(V) \cdot \text{Ind}(W)$

(5) $\text{Ind}(-V) = (-1)^{\dim M}(V)$

(6) If V has no zeros, then $\text{Ind}(V) = 0$

(7) $\text{Ind}(V) = \sum_C \text{Ind}_C(V)$ for all compact connected components C, assuming Z is the union of a finite number of compact connected components.

For certain vector fields the index is equal to classical invariants. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$. Let M be a compact n submanifold. Define V^f by $V^f(m) = f(m)$. If $f : \partial M \to \mathbb{R}^n - \vec{0}$, then

(8) Ind $V^f = \deg f$.

Suppose $f : U \to \mathbb{R}^n$ where U is an open set of $\mathbb{R}^n$. Let $V_f(m) = \vec{m} - \overrightarrow{f(m)}$. Then

(9) Ind $V_f =$ fixed point index of f on U.

Suppose $f : M \to N$ is a smooth map between two Riemannian manifolds. Let V be a vector field on N. Let f^*V be the pullback of V on M. We define the pullback by

$$\langle f^*V(m), \vec{v}_m\rangle = \langle V(f(m)), f_*(\vec{v}_m)\rangle.$$

Note that for $f : M \to \mathbb{R}$ and $V = \frac{d}{dt}$, we have $f^*V =$ gradient f.

Now suppose that $f : M^n \to \mathbb{R}^n$ where M^n is compact and V is a vector field on $\mathbb{R}^n$ so that f has no singular points near ∂M and V has no zeros on $f(\partial M)$. Then if $n > 1$

(10) $\text{Ind } f^*V = \sum w_i v_i + (\chi(M) - \deg \hat{N})$

where $\hat{N} : \partial M \to S^{n-1}$ is the Gauss map defined by the immersion of ∂M if $\mathbb{R}^n$ under f, and $v_i = \text{Ind}_{c_i}(V)$ for the i$^{\text{th}}$ zero of V and w_i is the winding number of the i$^{\text{th}}$ zero with respect to $f : \partial M \to \mathbb{R}^n$. That is calculated by sending a ray out from the i$^{\text{th}}$ zero and noting where it hits the immersed $n - 1$ manifold ∂M. At each point of intersection the ray is either passing inward or outward relative to the outward point normal N. Add up these point assigning $+1$ if the ray is going from inside to outside, and -1 if the ray goes from outside to inside.

§1. The Intermediate Value Theorem:

We have a map $f : I \to R$ so that $f(0) > 0$ and $f(1) < 0$. We must show that $f(c) = 0$ for some c. Consider V^f on I. $\text{Ind}(\partial_- V^f) = 2$. Hence (1) says Ind $V^f = -1$. Hence (6) implies that V^f has a zero, hence f hits 0.

§2. The Fundamental Theorem of Algebra:

The proof will also work for a polynomial in the quaternions, provided the homogeneous polynomial of top degree terms have an isolated zero. Let $f(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$. Consider V^f restricted to a disk about the origin with radius r so large that $|a_n z^n| > |a_{n-1}z^{n-1} + \cdots + a_0|$. Consider the homotopy $f(z)$ to $g(z) = a_n z^n$, via $f_t(z) = a_n z^n + t(a_{n-1}z^{n-1} + \cdots + a_0)$. The homotopy V^{f_t} of the associated vector fields is proper. No zero can appear on the boundary. Then homotopy $a_n z^n$ to z^n. Let $h(z) = z^n$, then contract D through smaller and smaller radii until $r = 1$. Then $\text{Ind}(V^h) = n \neq 0$ using (1) for example. Since the process is a proper otopy, by (2), $\text{Ind}(V^f) \neq 0$ and so by (6), V^f has a zero, so $f(z)$ has a root.

§3. Rouche's Theorem

If f is analytic on a region M so that $|f(z)| > |g(z)|$ for all $z \in \partial M$, then $f(z)$ has as many zeros as $f(z) - g(z)$, if the zeros are simple. The homotopy $f_t(z) = f(t) - tg(z)$ is proper, so Ind $V^f =$ Ind V^{f-g}. Now Ind $V^f =$ number of simple zeros.

§4. The Gauss-Lucas Theorem

This states that if $f(z)$ is a polynomial, the zeros of $f'(z)$ are contained in the convex hull of the zeros of $f(z)$.

First we study the case of a map $f : \mathbb{R}^m \to \mathbb{R}^n$ and a vector field V on $\mathbb{R}^n$. Suppose $\vec{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ represents a point and f_i and V_j are the components of f and V respectively.

THEOREM 1. $f^*V(\vec{x}) = (V_1(f(\vec{x})), \ldots, V_n(f(\vec{x}))\begin{pmatrix} \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_n}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_m} \end{pmatrix}$ *where* $\left(\frac{\partial f_i}{\partial x_j}\right)$ *is the Jacobian matrix.*

Proof. Let $\vec{\sigma}_x = (\sigma_1, \ldots, \sigma_m)$. Then the defining relation for $f^*V(m)$ given by $\langle f^*V(m), \vec{\sigma}\rangle = \langle V(f(m)), f_*(\vec{\sigma})\rangle$ yields

$$\langle f^*V(m), \vec{\sigma}\rangle = (V_1(f(m)), \ldots, V_n(f(m))\begin{pmatrix} \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_1}{\partial x_m} \\ \\ \frac{\partial f_n}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_m} \end{pmatrix}\begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_m \end{pmatrix}.$$

Hence the theorem is proved.

Now suppose $f : \mathbb{C} \to \mathbb{C}$. Then $f(z)$ can be written as $f(z) = u(z) + iv(z)$.

THEOREM 2. *If f is a complex analytic function and V is a continuous vector field on $\mathbb{C}$, then $f^*V(z) = \overline{f'(z)} \cdot V(f(z))$.*

Proof. Now $f'(z) = \frac{df(z)}{dz} = u_x + iv_x = v_y - iu_y$ since the Cauchy-Riemann equations hold if f is analytic. Hence $f'(z) = u_x(z) - iu_y(z)$. Now from Theorem 1

$$\begin{aligned} f^*V(z) &= (V_1(f(z)), V_2(f(z))\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\Big|_z \\ &= (V_1(f(z)), V_2(f(z))\begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix}\Big|_z \quad \text{by Cauchy Riemann} \\ &= (u_xV_1(f(z)) - u_yV_2(f(z)), \quad u_yV_1(f(z)) + u_xV_2(f(z)))\big|_z. \end{aligned}$$

In complex notation we may rewrite the equality above as

$$\begin{aligned} f^*V(z) &= [u_xV_1(f(z)) - u_yV_2(f(z))] + i[u_yV_1(f(z)) + u_xV_2(f(z))] \\ &= (u_x + iu_y)(V_1(f(z)) + iV_2(f(z))) \\ &= \overline{(u_x - iu_y)}(V_1(f(z)) + iV_2(f(z))) \\ &= \overline{f'(z)}V(f(z)). \end{aligned}$$

Thus $f^*V(z) = \overline{f'(z)}V(f(z))$. □

Now let V be the vector field given by $V(z) = z$. Then $f^*V(z) = \overline{f'(z)}f(z)$. Thus f^*V has zeros at the zeros of $f(z)$ and at the zeros of $f'(z)$. If $f(z)$ is a polynomial with zeros at the set $a_1, \dots, a_n$, then

$$f^*V(z) = \overline{f'(z)}f(z) = |f(z)|^2 \left(\sum_{i=1}^{n} \frac{z - a_i}{|z - a_i|^2} \right).$$

Now this vector field cannot have a zero outside of the convex hull of the zeros $a_1, \dots, a_n$. Thus the Gauss-Lucas Theorem holds.

Note that $W(z) = \frac{z}{|z|}$ is the pullback of grad$|z| = \nabla(|z|)$. Thus $f^*W = \nabla|f(z)|$. Now since W and V are both pointing in the same directions, we see that f^*V is orthogonal to the level curves $|f(z)| = k$. So if $M_k = \{z | f(z)| \leq k\}$, then

$$\begin{aligned}\chi(M_k) &= (\# \text{ of components of } M_k) - (\# \text{ of holes in } M_k)\\ &= (\# \text{ of zeros of } f(z)) - (\# \text{ of zeros of } f'(z) \text{ in } M_k).\end{aligned}$$

This is a theorem of Hurwitz.

§5. The Gauss-Bonnet Theorem

Equation (10) is a generalization of the extrinsic Gauss-Bonnet Theorem. We obtain a proof as follows. The Gauss-Bonnet Theorem as proved by Hopf goes as follows [HO_1] [S, P.386]. The curvature integrala of a closed submanifold ∂M of dimension $n-1$ in $\mathbb{R}^n$ is the degree of the Gauss map $\hat{N} : \partial M \to S^{n-1}$ where S^{n-1} is the unit sphere. Then Hopf generalized the Gauss-Bonnet Theorem by showing deg $\hat{N} = \frac{1}{2}\chi(\partial M)$ if n is odd. Now if we consider a map $f : M^n \to \mathbb{R}^n$ so that (10) holds, and if we let $x : \mathbb{R}^n \to \mathbb{R}$ be projection onto the x axis, then (10) becomes

(11) $\text{Ind } \nabla(f \circ x) = \chi(M) - \text{ deg } \hat{N}$

since $\nabla(f \circ x) = f^*(\nabla x)$ and ∇x has no zeros. Now for odd n, equation (5) states that $\text{Ind}(\nabla(f \circ (-x)) = -\text{Ind } \nabla(f \circ x)$. Hence $\chi(M) = \text{deg } \hat{N}$. But $\chi(M) = \frac{1}{2}\chi(\partial M)$. So the Gauss-Bonnet theorem is proven. Note that the argument for closed manifolds M of odd dimension have $\chi(M) = 0$ uses exactly the same path: For closed manifolds Ind $V = \chi(M)$ and by (5) $\text{Ind}(-V) = -\text{Ind}(V)$.

Also note the following 2 corollaries. (a) For n odd, Ind $\nabla(f \circ x) = 0$. (b) If f is an immersion, then deg $\hat{N} = \chi(M)$. This last is a theorem of Haefliger [Ha]. It follows since if f is an immersion, then $\nabla(f \circ x)$ has no zeros. Hence by (6) Ind $\nabla(f \circ x) = 0$ and so equation (11) yields deg $\hat{N} = \chi(M)$. [G_5].

§6. The Brouwer Fixed Point Theorem

Let $f : B \to B$ be a continuous map where B is a unit ball in $\mathbb{R}^n$. The Brouwer fixed point theorem asserts that f has a fixed point. We consider $f : B \to B \subset \mathbb{R}^n$. Let $V_f(m) = m - f(m)$ be the vector field.

The convexity of B implies V_f always points inside B for any m on the boundary. Hence $\partial_- B = \partial B$. So applying (1), we get

$$\text{Ind } V_f + \chi(S^{n-1}) = \chi(B),$$

hence

$$\text{Ind } V_f = 1 - (1 + (-1)^{n-1}) = (-1)^n \neq 0.$$

Hence V_f has a zero, hence f has a fixed point.

The Brouwer fixed point theorem can be greatly generalized. For example, let us say, that $f : M \to \mathbb{R}^n$, where M is a compact n-dimensional submanifold, is transversal to ∂M if the line segment from m to $f(m)$ is not tangent to ∂M at m for $m \in \partial M$. Then f has a fixed point if $\chi(M) - \sum \chi(\partial M_i) \neq 0$, when ∂M_i are the components of ∂M so that the line segment from m to $f(m)$ begins by entering M. [G_5].

§7. The Borsuk-Ulam Theorem

The key lemma, indeed many people call it the Borsuk-Ulam theorem itself, is that an odd map $f : S^n \to S^n$ has odd degree. Considering S^n as the unit sphere in $\mathbb{R}^{n+1}$, we say f is odd if $f(-x) = -f(x)$ for all $x \in S^n$.

Using the covering homotopy property for the covering space $S^n \xrightarrow{p} \mathbb{R}P^n$, we can homotopy f to an odd map f_1 so that there are only a finite number of pairs of Antipodal Points which are fixed by f_1. Now we extend $f_1 : S^n \to S^n$ to $g : B \to B$ by $f(r\vec{s}) = rf_1(\vec{s})$ where $\vec{s} \in \partial B$ and $r \in [0, 1]$. Then V^g is a vector field defined by $V^g(r\vec{s}) = g(r\vec{s}) = rf_1(\vec{s})$. Now Ind $V^g =$ deg g by (8) and deg $g =$ deg $f_1 =$ deg f. Hence (1) becomes deg $f +$Ind $\partial_- V^g = 1$. Now $\partial_- V^g$ has zeros exactly at those $m \in \partial B$ where $f_1(m) = -m$. But then $-m$ is also a zero of $\partial_- V^g$, and the symmetry of f_1 and the fact, from (1), that index only depends on pointing inside implies that the index at m is equal to the index at $-m$. Hence Ind $\partial_- V^g$ is even. Thus deg $f = 1 -$ Ind $\partial_- V^g$ is odd.

§8. The Jordan Curve Theorem

If $S^{n-1} \subset \mathbb{R}^n$, then $\mathbb{R}^n - S^n$ is split into two components, the inside and the outside. We will show that if $M^{n-1} \subset \mathbb{R}^n$, where is a smooth connected submanifold of $\mathbb{R}^n$, then $\mathbb{R}^n - M$ splits into the inside and the outside.

We choose a continuous normal vector field $\vec{N}$ on M. Let V be the Electric vector field generate by an electron e. Bring e so close to M that M looks like a hyperplane near e. Move e to the other side of the hyperplane at a speed near that of light along the normal direction. The vector field ∂V, which is V projected onto M, originally and finally has an isolated zero at the foot of the normal. Outside of a small ball about this zero the vector field ∂V does not change. Hence the index of the zero does not change. But the motion changes the zero from $\partial_+ V$ to $\partial_- V$ (or vice versa). Hence $\operatorname{Ind}(\partial_- V)$ changes by ± 1. Now suppose there were a path σ from the original position of e to the final position of e which does not cross M. Move e along σ. Then $\partial_- V$ undergoes an otopy which changes its index. Hence it is not a proper otopy by (2). Hence a zero must be on the frontier if $\partial_- V$ at some time. This can only happen when e is on M, contradicting the statement that σ avoids M.

Now M is the boundary between the component of $\mathbb{R}^n - M$ which "contains" ∞, and the bounded components. If we put e in one of the bounded components, then $\operatorname{Ind} V = 1$, since the electron is the only defect inside the union of the bounded components of $\mathbb{R}^n - M$. Call this union W. If we put e inside another component of W, then the index of this new V_1 is $+1$ also. Hence there is a proper homotopy between V and V_1. The set of defects of all the V_t contains the first and last position of the electron. So there is a small connected open set U containing c and e_1 which does not intersect M. So e and e' are in the same path component. So w is connected.

§9. Gottlieb's theorem

This theorem, named by Stallings in [St], has been considerably generalized, first by Rossett and then by Cheeger and Gromov. I hope the reader will forgive me for calling the theorem by my own name, but I wanted to prove theorems specified by short, commonly known names, using the Law of Vector Fields.

The key lemma in the original proof is: If X is a compact CW-complex so that $\chi(X) \neq 0$, then $G_1(X)$ is trivial. Then if X is a $\kappa(\pi, 1)$ we know that $G(x) =$ center of π. Thus we must show that if $F : X \times S^1 \to X$ so that $F|X \times *$ is the identity and if $\chi(X) \neq 0$, then $F|x \times S^1$ is homotopically trivial. The original proof used Nielsen-Wecken fixed point classes [G$_1$]. These were transformed by Stallings [St] into an algebraic setting which is of importance algebraically. The present proof is more elementary and does not need Nielsen-Wecken fixed point theory.

Suppose M is a regular neighborhood of an embedding of X in some $\mathbb{R}^n$. Then we have a map $F : M \times I \to M$ so that $F(m, 0) = m$ and $F(m, 1) = m$ for all $m \in M$. We may adjust F so that $F(m, t) \neq m$ for all $m \in \partial M$ and all t, and so that F_0 and F_1 are the identity outside a small collar neighborhood of the boundary.

Now we define the vector field T on $M \times I$ by $T(m, t) = (\vec{m} - \overrightarrow{F(m, t)}, t)$. Now T is a homotopy on $M \times I$. Let Z be the zeros of T on $M \times I$. Now Z is compact and $Z \cap (\partial M \times I)$ is empty. Let U be the open set of $M \times I$ so that $\|T(m, t)\| < \epsilon$ where ϵ is so small that two paths $\alpha(t)$ and $\beta(t)$ on M are homotopic if the distance between $\alpha(t)$ and $\beta(t)$ is always less than ϵ. Let W be a path component of U containing $m \times 0$. Then $T|U$ is a proper otopy from T_0 to T_1. Now $\operatorname{Ind} T_0 = \operatorname{Ind} T_1$ and $\operatorname{Ind} T_0 = \chi(M)$ and

Ind $T_1 = \chi(M)$. So the path connected W contains $M \times 0$ and $M \times 1$.

We may find a path $\gamma : I \to U$ so that $\gamma(0) = * \times 0$ and $\gamma(1) = * \times 1$. We can homotopy $F : M \times S^1 \to M$ to a G such that $G(\gamma(t), t) = \gamma(t)$. Then $\gamma \sim \alpha \cdot \gamma$ where $\alpha(t) = G(*, t)$. Hence $\alpha \sim 0$, which was to be shown.

§10. The Poincare-Hopf Theorem

If M is a close manifold and V is a continuous vector field defined entirely on M, then Ind $V = \chi(M)$. This is a special case of (1) because since the boundary ∂M is empty, so is $\partial_- V$ and so $\mathrm{Ind}(\partial_- V) = 0$.

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