INTERSECTION NUMBERS, TRANSFERS, AND GROUP ACTIONS

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Abstract

We introduce a new definition of *intersection number* by means of umkehr maps. This definition agrees with an obvious definition up to sign. It allows us to extend the concept parametrically to fibre bundles. For fibre bundles we can then define *intersection number transfers*. Fibre bundles arise naturally in equivariant situations, so the *trace of the action* divides intersection numbers. We apply this to equivariant projective varieties and other examples. Also we study actions of non-connected groups and their traces.

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1. Introduction

When two manifolds with complementary dimensions inside a third manifold intersect transversally, their intersection number is defined. This classical topological invariant has played an important role in topology and its applications. In this paper we introduce a new definition of intersection number.

The definition naturally extends to the case of parametrized manifolds over a base B; that is to fibre bundles over B . Then we can define transfers for the fibre bundles related to the intersection number. This is our main objective , and it is done in Theorem 6. The method of producing transfers depends upon the Key Lemma which we prove here. Special cases of this lemma played similar roles in the establishment of the Euler–Poincare transfer, the Lefschetz number transfer, and the Nakaoka transfer.

Two other features of our definition are: 1) We replace the idea of submanifolds with the idea of maps from manifolds, and in effect we are defining the intersection numbers of the maps; 2) We are not restricted to only two maps, we can define the intersection number for several maps, or submanifolds.

We can apply Theorem 6 to equivariant topology. When the maps involved are equivariant, or even if they are only *homotopy* equivariant, the intersection number is an equivariant invariant. This is seen by using the Borel construction to get the fibre bundle situation for which theorem 6 provides transfers. Transfers tell us much about equivariant topology. Much of this information can be concentrated in a single integer called the *trace of the action.*

The trace of an action is an integer which is defined by looking at all the possible transfers arising from the Borel construction. It was defined in [**G2**] and is very closely related to the exponent of an action as defined by W. Browder in [**B**]. In fact the trace equals the exponent for orientation preserving actions on manifolds. However the trace, in contrast to the exponent, is always defined in any equivariant setting. When the equivariant situation allows intersection numbers, the trace must divide the intersection number. This result is our main application of transfers to group actions and is found in Theorem 7 and is a consequence of Theorem 6.

In addition to showing that the trace divides intersection numbers, we examine how the trace behaves when the acting group is disconnected. In particular, we show that the trace of an action of a finite group G on a space X must be the product of the traces of the Sylow p-subgroups acting on X. This latter part is contained in the thesis of Murad Özaydin at Purdue University [**O**].

Finally, we give some examples of trace in action. In the case of nonsingular projective varieties, the degree of the variety is an intersection number. Thus in suitable equivariant situations the trace divides the degree of the variety. We obtain generalizations of a theorem of W. Browder and N. Katz [**B-K**] . Also we look at examples of elementary p–groups acting on products of spheres. In these situations we find that appropriate G–submanifolds imply that the trace divides associated determinants, permanents, and Pfaffians.

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2. Multiple functional intersection number

DEFINITION 1. Let $f_i : A_i \to M$ be a finite set of maps between closed oriented manifolds A_i and M. We suppose the sum of the co-dimensions of A_i equals the dimension of M. That means, assuming there are k maps f_i, that $\sum\limits_{i=1}^{k}(\dim M - \dim A_i) = \dim M$, or $\sum\limits_{i=1}^{k} \dim A_i = (k-1)\dim M$.

Next define $F : A_1 \times \cdots \times A_k \to M \times \cdots \times M$ by $F = f_1 \times \cdots \times f_k$. Define the <u>multiple intersection number</u>, denoted $f_1 \bullet f_2 \bullet \cdots \bullet f_k$, as the integer

$$[\Delta^* F^!(1)] \cap [M] \in H_0(M;\mathbb{Z}) \cong \mathbb{Z}.$$

Here $\Delta : M \to M \times \cdots \times M$ is the diagonal map, $F^!$ denotes the Umkehr map on cohomology and $1 \in H^0(A_1 \times \cdots \times A_k)$. We denote by $[M]$ the orientation class of M. We require M to be connected, so that $H_0(M;\mathbb{Z}) \cong \mathbb{Z}$, but the A_i need not be connected. So the $1 \in H^0(A_1 \times \cdots \times A_k)$ is the sum of the generators
$1 \in H^0$ (connected components of $A_1 \times \cdots \times A_n) \cong \mathbb{Z}$, and each of those generators is equal to a product $1_A \times 1_B \times \ldots$, where the $A, B, \ldots$ are connected components of $A_1, A_2, \ldots$ respectively. The Umkehr map $F^!$ is defined as the composition $D^{-1}_{M^\times} F D_{A^\times}$ where D_M denotes the Poincare duality map $H^*(M;\mathbb{Z}) \xrightarrow{\cong} H_*(M;\mathbb{Z})$ given by $D_M(x) = x \cap [M]$. Here $M^\times = M \times \cdots \times M$ where the product is taken k-times and $A^\times = A_1 \times \cdots \times A_k$.

Obviously the multiple intersection number depends on the choices of orientation on $A^{\times}$ and on M. For the case of two connected closed oriented submanifolds A and B of complimentary dimensions in M, the usual intersection number $A \bullet B$ agrees with $f \bullet g$ (up to sign) where f and g are the inclusion maps. That is because the usual intersection number ends up being given by the cup product of the Poincare duals of the fundamental classes $[A]$ and $[B]$ in $H_*(M;\mathbb{Z})$. Using the concept of the Umkehr map, this is just $f^!(1_A) \cup g^!(1_B)$ which the following lemma implies is equal to $\pm(f \bullet g)[M]$.

REMARK: The reader may ask why the multiple functional intersection number $f_1 \bullet f_2 \bullet \cdots \bullet f_k$ is defined using Umkehr maps instead of the obvious definition as the cup product of the Poincare duals of the homology classes $f_{i*}[A_i]$ capped with $[M]$? As one can see from Lemma 3, the obvious definition differs from our definition by a complicated sign. The reason we need Umkehr maps is to construct the transfer in Theorem 6. Note that the Key Lemma cannot be stated without Umkehr maps. In general, if one can express an invariant in terms of Umkehr maps, then there is a good chance transfers can be defined. This is especially true since the Key Lemma holds. The sign difference in Lemma 3 is not important; it is just there.

LEMMA 2. *$\Delta^*(f \times g)^!(1_A \times 1_B) = (-1)^{nq} f^!(1_A) \cup g^!(1_B)$ where $q = n - \dim B$ and $n = \dim M$.*

Proof:

$$\begin{aligned}
&\Delta^*(f \times g)^!(1_A \times 1_B)\\
&\quad = \Delta^* D^{-1}_{M\times M}(f_*([A]) \times g_*([B]))\\
&\quad = \Delta^* D^{-1}_{M\times M}((f^!(1_A) \cap [M]) \times (g^!(1_B) \cap [M])))\\
&\quad = (-1)^{nq} \Delta^* D^{-1}_{M\times M}((f^!(1_A) \times g^!(1_B)) \cap ([M] \times [M]))\\
&\quad = (-1)^{nq} \Delta^*(f^!(1) \times g^!(1)) = (-1)^{nq} f^!(1) \cup g^!(1).
\end{aligned}$$

In fact, we have in general

LEMMA 3. *$\Delta^* F^!(1) = (-1)^a f_i^!(1_{A_1}) \cup \cdots \cup f_k^!(1_{A_k})$ where $a = n[(k-1)q_k + (k-2)q_{k-1} + \cdots + q_2]$ and $q_i = n - \dim A_i$, the codimension of A_i.*

Proof: The same as Lemma 2, only the notation is much more involved. Counting the sign change is the only complication.

Next we extend the definition of multiple intersection number to the case where M is a connected oriented manifold with boundary ∂M and the $f_i : A_i \to M$ are maps of pairs $(A_i, \partial A_i) \xrightarrow{f_i} (M, \partial M)$. We assume that at least one of the A_i has an empty boundary. Also the A_i are all oriented and the sum of the codimension $(n - \dim A_i)$ is equal to n as before. We let DM denote the double of M. It is a closed oriented manifold. We let $Df_i : DA_i \to DM$ be the double of the map $f_i : (A_i, \partial A_i) \to (M, \partial M)$ in the case where A_i has a boundary, and if $\partial A_i = \phi$ we let $Df_i : A_i \xrightarrow{f_i} M \xrightarrow{j} DM$ be the composition of f_i with the standard inclusion j of M into its double.

DEFINITION 4. We define the multiple intersection number $f_1 \bullet f_2 \bullet \cdots \bullet f_k$ to be the intersection number $Df_1 \bullet Df_2 \bullet \cdots \bullet Df_k$.

REMARKS: a) We required at least one of the A_i to have an empty boundary. We can define the intersection number as above in the case when all the A_i have non-empty boundaries, but in this case it will always be zero.

b) The intersection number is obviously preserved under homotopy of the f_i.

c) If $f_1 \bullet \cdots \bullet f_k \neq 0$, then the intersection of the images of the f_i is not empty, that is $\bigcap\limits_{i=1}^{k} f_i(A_i) \neq \phi$.

d) A geometric interpretation of multiple intersection number for the case when the A_i are submanifolds of M should go as follows: choose orientations of the normal bundles ν_i consistent with the choices of orientation of M and A_i. Assuming the A_i are mutually transversal at the points $x_j \in \bigcap\limits_{i=1}^{k} A_i$, assign to the point x_j the number $+1$ if the orientations at x_j combine to give the orientation of M at x_j, and assign x_j the number -1 otherwise. The sum of these local multiple intersection numbers is the global intersection number.

e) Let ν^k be a k bundle over the closed oriented manifold M^n. If $n = kr$, the self-intersection number of the zero section s is defined to be $s \bullet \cdots \bullet s$. Let $\chi(\nu)$ denote the Euler class of a bundle ν. Then

$$\chi(\nu \oplus \cdots \oplus \nu) = [\chi(\nu)]^r = \pm (s \bullet \cdots \bullet s)[\overline{M}] \in H^n(M; \mathbb{Z})$$

An application of the idea of intersection number can be used in a dual situation to calculate the degree of a map. Suppose that M is a closed smooth oriented connected manifold and suppose that we have a family of k smooth maps $f_i : M \to A_i$ where the A_i are connected smooth closed oriented manifolds the sum of whose dimensions equals the dimension of M. The relevant invariant, a "co-intersection number", is the degree of the composition

$$G : M \xrightarrow{\Delta} M \times \cdots \times M \xrightarrow{f_1 \times \cdots \times f_k} A_1 \times \cdots \times A_k.$$

COROLLARY 5. *With the notation of the preceding paragraph,* $\pm \deg G = i_1 \bullet \cdots \bullet i_k$ *where* $i_j : f_j^{-1}(a_j) \to M$ *and* a_j *is a regular value of* f_j *in* A_j*. Thus the degree of* G *is the intersection number of the fibres of the* f_i*.*

Proof: For any smooth map $f : M \to N$ between closed oriented manifolds and fibre $F = f^{-1}(b)$ where b is a regular point, we have $i_*[F] = f^*([\overline{N}]) \cap [M]$ where $i : F \to M$ is the inclusion, [**G1**]. So, in other words, $i^!(1_F) = f^*([\overline{N}])$. Now $(\deg G)[\overline{M}] = G^*([\overline{A}_1] \times \cdots \times [\overline{A}_k]) = f_1^*([\overline{A}_1]) \cup \cdots \cup f_k^*([\overline{A}_k]) = i_1^!(1_{F_1}) \cup \cdots \cup i_k^!(1_{F_k}) = \pm(i_1 \bullet \cdots \bullet i_k)[\overline{M}]$.

3. Transfers and intersection numbers

Consider the following commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{f} & F_1 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i_1} \\ E & \xrightarrow{\overline{f}} & E_1 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_1} \\ B & \xrightarrow[1_B]{} & B \end{array}$$

Here p and p_1 are fibre bundle projections and i and i_1 are inclusion of a fibre. The map $\overline{f}$ is a fibre preserving map which restricts to f on the fibre. In addition every space is a closed oriented manifold

KEY LEMMA. $f^! i^* = (-1)^{B(F_1 - F)} i_1^* \overline{f}^!$ *where* $(-1)^X$ *means* $(-1)^{\dim X}$.

Proof: Let U be a small ball in B centered on the base point $*$ where $p^{-1}(*) = F$. Then $V = p^{-1}(U)$ and $V_1 = p_1^{-1}(U)$ are open sets of E and E_1 respectively.

The following diagram commutes except for the top rectangle. The undecorated arrows are induced by inclusions.

$$
\begin{tikzcd}
 & H^*(F) & f^! & & H^*(F) \\
 & H^*(\overline{U}\times F) \arrow[u] \arrow[r, "\cap[\overline{U},\dot{U}]\times[F]", "\cong"'] & H_*((\overline{U},\dot{U})\times F) \arrow[r, "(1\times f)_*"] \arrow[d, "ex", "\cong"'] & H_*((\overline{U},\dot{U})\times F_1) \arrow[d, "ex", "\cong"'] & H^*(\overline{U}\times F_1) \arrow[u] \arrow[l, "\cap[\overline{U},\dot{U}]\times[F_1]"', "\cong"] \\
 & H^*(E) \arrow[u, "i^*"] \arrow[r, "\cap[E,E-V]"] & H_*(E,E-V) \arrow[r, "\overline{f}_*"] & H_*(E_1,E_1-V_1) & H^*(E_1) \arrow[u, "i_1^*"] \arrow[l, "\cap[E_1,E-V_1]"'] \\
\overline{f}^! : & H^*(E) \arrow[u, "1_E", "\cong"'] \arrow[r, "\cap[E]", "\cong"'] & H_*(E) \arrow[u] \arrow[r, "\overline{f}_*"] & H_*(E_1) \arrow[u] & H^*(E_1) \arrow[u, "1_{E_1}", "\cong"'] \arrow[l, "\cap[E_1]"', "\cong"]
\end{tikzcd}
$$

Now the top rectangle commutes up to $(-1)^{B(F_1-F)}$, since the second row gives the following

$$\begin{aligned}
&1 \times x \mapsto (1 \times x) \cap [\overline{U}, \dot{U}] \times [F] = (-1)^{xB}[\overline{U}, \dot{U}] \times (x \cap [F]) \\
&\mapsto (-1)^{xB}[\overline{U}, \dot{U}] \times f_*(x \cap [F]) = (-1)^{xB}[\overline{U}, \dot{U}] \times (f^!(x) \cap [F_1]) \\
&= (-1)^{xB+(x+B(F_1-F))B}(1 \times f^!(x)) \cap ([\overline{U}, \dot{U}] \times [F_1]) \\
&\mapsto (-1)^{B(F_1-F)} 1 \times f^!(x)
\end{aligned}$$

This proves the lemma.

Suppose we have the following k fibre bundle squares.

$$\begin{tikzcd}
A_i \arrow[r, "f_i"] \arrow[d, "i_i"] & M \arrow[d, "i"] \\
E_i \arrow[r, "\widetilde{f}_i"] \arrow[d, "p_i"] & E \arrow[d, "p"] & i = 1, \ldots, k \\
B \arrow[r, "1_B"'] & B
\end{tikzcd}$$

where each space is a closed oriented manifold, M and B are connected and the codimension of the A_i sum to the dimension of M. More generally we may assume that M and B and hence E have non-empty boundaries and that some of the E_i have non-empty boundaries and in those cases $\widetilde{f}_i : (E_i, \partial E_i) \to (E, \partial E)$. Of course all spaces are compact.

THEOREM 6. *There is a transfer associated to the fibre bundle p with trace $(f_1 \bullet \cdots \bullet f_k)$. That is, there is a homomorphism $\tau_* : H_*(B, G) \to H_*(E; G)$ so that $p_* \circ \tau_*$ is multiplication by $(f_1 \bullet \cdots \bullet f_k)$. Similarly there is a homomorphism $\tau^* : H^*(E, G) \to H^*(B, G)$ so that $\tau^* \circ p^*$ is multiplication by $(f_1 \bullet \cdots \bullet f_k)$.*

Proof: Let $E_1 \circ \cdots \circ E_k$ denote the multiple fibre product of the fibre bundles $p_i : E_i \to B$. That is,

$$E_1 \circ \cdots \circ E_k = \{(e_1, \ldots, e_k) \in E_1 \times \cdots \times E_k | p_1(e_1) = \cdots = p_k(e_k)\}$$

Now the map $\overline{p} : E_1 \circ \cdots \circ E_k \to B$ defined by $p(e_1, \ldots, e_k) = p(e_1) \in B$ is a fibre bundle projection with fibre $A_1 \times \cdots \times A_k$. Also we have the multiple fibre product

$E \circ \cdots \circ E \to B$ with fibre $M \times \cdots \times M$. Define $\widetilde{F} : E_1 \circ \cdots \circ E_k \to E \circ \cdots \circ E$ by $\widetilde{F}(e_1, \ldots, e_k) = (\widetilde{f}_1(e_1), \ldots, \widetilde{f}_k(e_k))$. The diagonal $\widetilde{\Delta} : E \to E \circ \cdots \circ E$ given by $\widetilde{\Delta}(e) = (e, \ldots, e)$. All this fits into a commutative diagram

$$\begin{array}{ccccc} A_1 \times \cdots \times A_k & \xrightarrow{F} & M \times \cdots \times M & \xleftarrow{\Delta} & M \\ \downarrow k & & \downarrow j & & \downarrow i \\ E_1 \circ \cdots \circ E_k & \xrightarrow{\widetilde{F}} & E \circ \cdots \circ E & \xleftarrow{\widetilde{\Delta}} & E \\ \downarrow & & \downarrow & & \downarrow p \\ B & \xrightarrow{1_B} & B & \xleftarrow{1_B} & B \end{array}$$

Now we can define the class $\widetilde{\Delta}^* \widetilde{F}^!(1) \in H^n(E; G)$ in the case of closed E. Then $i^*(\widetilde{\Delta}^* \widetilde{F}^!(1)) = \Delta^* j^* \widetilde{F}^!(1) = \pm \Delta^* F^! k^*(1) = \pm \Delta^* F^!(1) = \pm (f_1 \bullet \cdots \bullet f_k)[\overline{M}]$. The equalities are true by commutativity and the key lemma and the last equality is the definition. The transfer maps are defined as usual: $\tau^*(x) = p^!(x \cup \widetilde{\Delta}^* \widetilde{F}^!(1))$ and $\tau_*(x) = p_!(\widetilde{\Delta}^* \widetilde{F}^!(1) \cap x)$.

In the case that M has a non-empty boundary, and B is closed, taking the doubles $D\widetilde{f}_i : DE_i \to DE$ and $D\widetilde{f}_j : E_{j_i} \to E \to DE$ gives a transfer for $DM \to DE \to B$. Since $M \to E \to B$ is a fibrewise retraction $DM \to DE \to B$, the transfer defined for the latter restricts to the former with the same trace, $f_1 \bullet \cdots \bullet f_k$. In the remaining case where B has non-empty boundary, we consider the pullback of $M \to E \xrightarrow{p} B$ by the retraction map $r : DB \to B$. Suppose this pullback is denoted $M \to DE \xrightarrow{Dp} DB$. Then Dp admits a transfer with associated number $f_1 \bullet \cdots \bullet f_k$, and so p, which is a fibrewise retract of Dp, admits a transfer with associated number $f_1 \bullet \cdots \bullet f_k$.

4. The trace of an action

Suppose G is a group acting on spaces A_i and M and suppose $f_i : A_i \to M$ are homotopy G-maps. Let us say that $i : W \to V$ is a *homotopy G-map* if there is a fibre bundle square

$$\begin{array}{ccc} W & \xrightarrow[i]{} & V \\ \downarrow & & \downarrow \\ E & \xrightarrow[f]{} & V_G \\ \downarrow & & \downarrow \\ B_G & \xrightarrow{1} & B_G \end{array}$$

which extends i to a map f over 1_{B_G}. Here V_G is the Borel construction $E_G \times_G V$

In [**G2**] and [**O**] we introduced the integer $\mathrm{tr}(p)$, read the trace the fibration p. Let $\deg(p)$ be the smallest positive integer N so that there is a homomorphism τ so that $p_* \circ \tau =$ multiplication by N. If no such integer exists then set $\deg(p)$ equal to zero. In fact, $\deg(p)$ is the greatest common divisor of these numbers. Next the *trace* of p, denoted $\mathrm{tr}(p)$, is the least upper bound of all the positive $\deg(p_f)$ where the p_f are the pullbacks of p by arbitrary maps $f : X \to B$. If the least upper bound does not exist set $\mathrm{tr}(p)$ equal to zero. In fact the $\mathrm{tr}(p)$ is the least common multiple of these numbers. Then we define the trace of an action, denoted $\mathrm{tr}(G, M)$, by $\mathrm{tr}(G, M) = \mathrm{tr}(p_G)$ where $p_G : E_G \times_G M \to B_G$ is the Borel construction of the universal G-principal bundle $G \to E_G \to B_G$ with M.

THEOREM 7. *Suppose G is a group so that the classifying space B_G has finite type and suppose that M is an oriented compact G-manifold and $f_i : A_i \to M$ are homotopy G-maps from oriented compact manifolds A_i which preserve boundaries. Then* $\mathrm{tr}(G, M) | (f_1 \bullet \cdots \bullet f_n)$.

Proof: Let B be a regular neighborhood of the $(\mathrm{Dim}\, M + 2)$- skeleton embedded in some Euclidean space. So B is an oriented manifold with boundary. Let $M \to E \xrightarrow{p} B$ be the pullback of $M \to M_G \to B_G$ onto B where the map $B \to B_G$, which induces the pullback, is homotopic to the inclusion of the $(\mathrm{Dim}\, M + 2)$-skeleton. By Proposition (6.6) of [**G2**] we have $\mathrm{tr}(G, M) = \deg(p)$. By theorem 6 and the definition of $\deg p$, we see that $f_1 \bullet \cdots \bullet f_k | \deg(p) = \mathrm{tr}(p) = \mathrm{tr}(G, M)$.

Theorem 7 allows us to add multiple intersection numbers to the list of topological and group theoretical properties divided by the trace of an action. Most of this list so far may be found in Theorem (1.1) of [**G2**].

REMARK: By Theorem (1.5) of [**G2**], if G acts orientably on a closed manifold M^n the trace $\mathrm{tr}(G, M)$ is the order of quotient group of the top dimensional integer cohomology group divided by the image of fibre inclusion of the cohomology of $H^n(EG \times_G M)$, if this is finite, and zero otherwise. This is just what Browder's exponent is in [**B**]. In this special case the trace appears more tractible than in the general definition. But the general definition works for any action and therefore appears in theorems which are not restricted to closed manifolds and orientation preserving actions. The list of properties of the general trace is quite substantial and they work together well, so that despite its strange looking definition it is actually quite easy to use the concept. This is exemplified by the arguments in the next section which do not use the exponent definition above and so result in facts which are true in more generality and which yield even more additions to the list of properties of the trace.

5. Traces and subgroups

Let G be a group, X a G-space and let H be a subgroup of G. By ([**G2**], Theorem 6.2a), $\mathrm{tr}(H,X)$ divides $\mathrm{tr}(G,X)$. We will now show that the quotient divides $\mathrm{tr}(G,G/H)$ where G/H is the G space consisting of a single orbit of isotropy type H.

PROPOSITION 8. *Let H be a subgroup of G, and let X be a G-space. Then*

$$\mathrm{tr}(H,X) \,\big|\, \mathrm{tr}(G,X) \,\big|\, \mathrm{tr}(G,G/H)\,\mathrm{tr}(H,X).$$

Proof: We already know the left division by [**G2**], Theorem 6.2a. For the second, given any map $f: Y \to B_G$, let $\widetilde{Y}$ be the pullback fitting into the diagram:

$$\begin{tikzcd}
G/H \arrow[r, equal] \arrow[d] & G/H \arrow[d] \\
\widetilde{Y} \arrow[r, "\widetilde{f}"] \arrow[d, "q"] & (G/H)_G = B_H \arrow[d] \\
Y \arrow[r, "f"] & B_G.
\end{tikzcd}$$

From the definition of $\mathrm{tr}(G,G/H)$ as the l.c.m. of the degrees of the projection maps over all pullbacks of the Borel construction, it follows that $\deg(\widetilde{Y} \to Y)$ divides $\mathrm{tr}(G,G/H)$. Hence there is a transfer $\tau : H_*(Y;\mathbb{Z}) \to H_*(\widetilde{Y};\mathbb{Z})$ so that $q_* \circ \tau = \mathrm{tr}(G,G/H)$.

From the diagram

$$\begin{tikzcd}
 & X \arrow[r, equal] \arrow[d] & X \arrow[d] \\
G/H \arrow[r] \arrow[d, equal] & X_H \arrow[r] \arrow[d] & X_G \arrow[d] \\
G/H \arrow[r] & B_H \arrow[r] & B_G
\end{tikzcd}$$

we get the pullback diagram (via f and $\widetilde{f}$)

$$\begin{tikzcd}
 & X \arrow[r, equal] \arrow[d] & X \arrow[d] \\
G/H \arrow[r] \arrow[d, equal] & \widetilde{f}^*(X_H) \arrow[r, "\widetilde{q}"] \arrow[d, "\widetilde{p}"] & f^*(X_G) \arrow[d, "p"] \\
G/H \arrow[r] & \widetilde{Y} \arrow[r, "q"] & Y
\end{tikzcd}$$

If σ is a transfer realizing $\deg(\widetilde{p})$ then $\rho = \widetilde{q}_* \circ \sigma \circ \ \tau$ satisfies $p_*\rho = p_* \circ \widetilde{q}_* \circ \sigma \circ \tau = q_* \circ \widetilde{p}_* \circ \sigma \circ \tau = \operatorname{tr}(G, G/H) \deg(\widetilde{p})$. Hence $\deg(p) \big| \operatorname{tr}(G, G/H) \deg(\widetilde{p})$. Now taking the lowest common multiple over all $f : Y \to B_G$, and noting that $\operatorname{tr}(H, X)$ is defined as the l.c.m. of degrees over a set containing all $\widetilde{f} : Y \to B_H$, we get $\operatorname{tr}(G, X)$ divides $\operatorname{tr}(G, G/H) \operatorname{tr}(H, X)$. $\square$

The G space G/H consisting of a single orbit is perhaps the simplest case we should consider, also important in view of the proposition above. In general, with some finiteness assumptions, we have $\operatorname{tr}(G, G/H)$ dividing the Euler characteristic $\chi(G/H)$ ([**G2**], Proposition 6.7b). When G/H is finite we have $\operatorname{tr}(G, G/H)$ equalling $\chi(G/H) = [G : H]$, but $\operatorname{tr}(G, G/H)$ does not equal $\chi(G/H)$ in general.

LEMMA 9. *Let H be a subgroup of G such that G/H is finite and discrete. Then* $\operatorname{tr}(G, G/H) = [G : H]$.

Proof: We have $\operatorname{tr}(G, G/H) | \chi(G/H) = [G : H]$. Hence it will suffice to find a pullback of the Borel construction $G/H \to B_H = (G/H)_G \to B_G$ whose projection has degree $[G : H]$. Let $\sigma_1, \ldots, \sigma_n$ be loops representing classes in $\pi_1(B_G)$ mapping onto $\pi_0(G/H) = G/H$ in the homotopy exact sequence

$$1 \to \pi_1(B_H) \to \pi_1(BG) \to G/H \to *.$$

Let $r : M \to \overset{i}{\vee} S^1$ be a retraction of a closed, connected, oriented manifold on a bouquet of n circles. Let f be given by

$$f : M \xrightarrow{r} \overset{i}{\vee} S^1 \xrightarrow{\vee \sigma} B_G.$$

Then $f^*(B_H)$ is a closed, connected, oriented manifold, covering M with fiber G/H. Thus the projection $\rho : f^*(B_H) \to M$ has degree exactly $[G : H]$. □

Combining Propositions 8 and 9 we obtain the following.

COROLLARY 10. *If G/H is finite then*

$$\operatorname{tr}(H, X) \big| \operatorname{tr}(G, X) \big| [G : H] \operatorname{tr}(H, X)$$

THEOREM 11. *Let G be a finite group and let X be a G space. Then*

$$\operatorname{tr}(G, X) = \Pi \ \operatorname{tr}(G_p, X)$$

where G_p is a Sylow p-subgroup of G and the product is over all primes dividing the order of G.

Proof: Since $\operatorname{tr}(G_p, X) \big| \ |G_p|$ and $|G_p|$ is relatively prime to $[G : G_p]$ we see that the divisions

$$\operatorname{tr}(G_p, X) \big| \operatorname{tr}(G, X) \big| [G : G_p] \operatorname{tr}(G_p, X)$$

implies that the p-primary factor of $\operatorname{tr}(G, X)$ is given by $\operatorname{tr}(G_p, X)$.

In the proof above, the relevant fact about a collection G_p was that their orders are relatively prime to each other and the order of G is the products of the orders of the G_p. Hence if $\{H_i\}$ is a collection of subgroups satisfying this condition, we have

$$\operatorname{tr}(G, X) = \prod_i \operatorname{tr}(H_i, X)$$

Now we consider the consequences of Proposition 8 for a compact Lie group G. We will denote the connected component of the identity by G_0, a maximal torus of G_0 by T, the normalizer of T in G_0 by $N(T)$, and the Weyl group $N(T)/T$ by W.

LEMMA 12. *Let X be a G-space (G compact Lie). Then $\operatorname{tr}(G, X)$ divides $[G : G_0] \operatorname{tr}(N(T), X)$. In particular if G is connected then $\operatorname{tr}(G, X)$ equals $\operatorname{tr}(N(T), X)$.*

Proof: Note that $\operatorname{tr}(G_0, G_0/N(T)) \big| \chi(G_0/N(T))$ by ([**G2**], 6.7b). But it is known that $\chi(G_0/N(T)) = 1$, hence $\operatorname{tr}(G_0/N(T)) = 1$ thus Proposition 8 tells us that

$$\operatorname{tr}(N(T), X) \big| \operatorname{tr}(G_0, X) \big| \operatorname{tr}(N(T), X)$$

Hence $\operatorname{tr}(N(T), X) = \operatorname{tr}(G_0, X)$. Now Corollary 10 gives us $\operatorname{tr}(G, X) \big| [G : G_0] \operatorname{tr}(G_0, X) = [G : G_0] \operatorname{tr}(N(T), X)$.

Now Lemma 12 gives us $\operatorname{tr}(N(T), X) \big| |W| \operatorname{tr}(T, X)$. When X is a compact manifold M, we know from ([**G2**], 6.11) that $\operatorname{tr}(T, M) = 1$, if the action has a stationary point; or $\operatorname{tr}(T, M) = 0$, if the action does not have a stationary point. Thus we obtain

COROLLARY 13. *Let G be a compact Lie group acting on a compact manifold M. Then $\operatorname{tr}(G, M)$ is either zero, or it divides $[G, G_0]|W|$ where W is the Weyl group of G_0.*

THEOREM 14. *Let G be a compact Lie Group acting on a compact manifold M. Let W_p be the Sylow p-subgroups of the Weyl group W, and let $\hat{W}_p$ is the pre-image in $N(T)$ of W_p in $W = N(T)/T$. Then*

$$\operatorname{tr}(G, M) = \prod_p \operatorname{tr}(\hat{W}_p, M)$$

where the product is taken over all the p dividing $|W|$.

Proof: We may assume that $\operatorname{tr}(\hat{W}_p, M) \neq 0$ because otherwise $\operatorname{tr}(T, M) = 0$, which implies $\operatorname{tr}(G, M) = 0$; then both sides of the equation would be zero, proving the theorem. So since $\operatorname{tr}(\hat{W}_p, M) \neq 0$, we have $\operatorname{tr}(T, M) = 1$, so from Proposition 8, $\operatorname{tr}(\hat{W}_p, M) \big| \, |\hat{W}_p|$. Now $\operatorname{tr}(\hat{W}_p, M) \big| \operatorname{tr}(N(T), M) \big| [W : W_p] \operatorname{tr}(\hat{W}_p, M)$ by ([**G2**], 6.2.b) for the first division and by Corollary 10 for the second division. Hence $\operatorname{tr}(\hat{W}_p, M)$ is a common divisor of $|W_p|$ and $\operatorname{tr}(G, M) = \operatorname{tr}(N(T), M)$, and in fact is the greatest common divisor since $\operatorname{tr}(G, M)/\operatorname{tr}(\hat{W}_p, M)$ is not a multiple of p. Since $\operatorname{tr}(G, M)$ divides $|W| = \prod_p W_p$, the theorem follows.

REMARKS: a) Theorem 14 can be generalized for a family $\{H_i\}$ of subgroups of W such that $\Pi|H_i| = |W|$ and the $|H_i|$ are pairwise relatively prime.

b) The following example shows that $\operatorname{tr}(G, G/H)$ does not always equal $\chi(G/H)$, in fact it shows that $\operatorname{tr}(G, G/T) \neq \chi(G/T) = |W|$.

(c) In fact $\operatorname{tr}(S^3, M)$ is always zero or 1 as M is a compact manifold. This follows from the following example where $\operatorname{tr}(S^3, S^3/S^1) = 1$. Hence $\operatorname{tr}(S^3, M) = \operatorname{tr}(S^1, M)$

EXAMPLE. Let $G = S^3$ and $H = S^1$. Then $\operatorname{tr}(S^3, S^3/S^1) = 1$ but $\chi(S^3/S^1) = 2$.

Proof: This can be seen by considering the fibration $S^2 \xrightarrow{i} B_{S^1} \xrightarrow{p} B_{S^3}$. This is a fibration which arises from the action of S^3 on $S^2 = S^3/S^1$ by the Borel construction. Now $H_i(B_{S^3}) \cong 0$ for $0 < i < 4$, so i^* is an isomorphism $i^* : H^2(B_{S^2}) \to H^2(S^2)$. Thus the generator of $H^2(S^2)$ is in the image of i^*. Hence by theorem 1.5 of [**G2**] we see that $\mathrm{tr}(S^3, S^3/S^1) = 1$.

6. Applications of the trace theorems

An application of Theorem 11 gives an extension of a result of Browder and Katz [**B-K**] to the nonfree case. Recall that a nonsingular complex projective variety V of dimension k is a closed, complex analytic submanifold of a complex projective space P^n, given as the zero set of a set of homogeneous polynomials (in $n+1$ variables). The Chern class of the canonical line bundle $c(\xi)$ generates the cohomology of P^n. The hyperplane section α in $H^2(V;\mathbb{Z})$ is the image of $c(\xi)$ under the map induced by inclusion $i^* : H^2(P^n;\mathbb{Z}) \to H^2(V;\mathbb{Z})$. We also have $\alpha^k = \deg(V)\,[\overline{V}]$ where $[\overline{V}]$ is the fundamental class in cohomology. The degree of the variety V, $\deg(V)$ (not to be confused with the degree of a map), depends on the particular embedding of V in the projective space, since the hyperplane section α depends on the embedding.

Proposition 15. *Let G be a finite group acting on a nonsingular complex projective variety V. Assume $H^1(V) = 0$ and the hyperplane section α in $H^2(V)$ is left invariant by G. Then*

$$\mathrm{tr}(G,V)^2 \big| \; |G| \; (\deg V)^2.$$

If all Sylow subgroups of G are cyclic, then $\mathrm{tr}(G,V)$ *divides* $\deg V$.

Proof: The p-primary part of $\mathrm{tr}(G,V)$ is exactly $\mathrm{tr}(G_p,V)$ for the Sylow p-subgroup G_p. Hence we may assume that G is a p-group. The action of G is orientation preserving because α and thus $\alpha^k = (\deg\ V)[\overline{V}]$ is invariant ($k =$ dimension of V). In the cohomology Serre spectral sequence of the Borel construction $V \to V_G \to B_G$ we have $E_2^{0,2} \cong H^2(V)^G$ and $E_2^* \cong H^*(G)$. These survive to E_3 since $H^1(V)$ is zero. Let $d\alpha \epsilon H^3(G)$ be the image of α under

$$d : E_3^{0,2} \cong H^2(V)^G \to H^3(G) \cong E_3^{3,0}.$$

Now there is a subgroup K of G such that $[G:K]^2$ divides $|G|$ and $d\ \alpha$ maps to 0 under the restriction map $H^3(G) \to H^3(K)$ ([**B-K**], 2.1). Restricting the action to K, from the functoriality of the Serre spectral sequence we deduce that α survives to E_∞ (for $V \to V_K \to B_K$). Hence α and $\alpha^k = \deg V[\overline{V}]$ are in the image of $H^{2k}(V_K)$ and thus $\mathrm{tr}(K,V)$ divides $\deg V$. Then $\mathrm{tr}(G,V)\big|[G:K]\,\mathrm{tr}(K,V)|[G:K]\deg V$, and squaring the first and last terms, $\mathrm{tr}(G,V)^2\big|[G:K]^2(\deg V)^2\big|\ |G|(\deg V)^2$. When G is cyclic, $H^3(G)$ is zero so $d\alpha$ is zero. We don't have to restrict to K, and we conclude $\mathrm{tr}(G,V)\big|\deg V$. $\square$

When G acts freely, $\mathrm{tr}(G,V) = |G|$ and we have $|G|$ divides $(\deg V)^2$ as in [**B-K**].

Now the degree of V is just the self intersection number of W with itself where $W = V \cap P^{k-1}$. So if G is acting on V such that W is left invariant, then by Theorem 7 we have $\mathrm{tr}(G,V)\big|\deg(V)$. In fact we can relax the condition that the inclusion $i : W \to V$ is a G-map. Recall that $i : W \to V$ is a homotopy G-map if there is a fibre bundle square

$$\begin{array}{ccc} W & \xrightarrow[i]{} & V \\ \downarrow & & \downarrow \\ E & \xrightarrow[f]{} & V_G \\ \downarrow & & \downarrow \\ B_G & \xrightarrow{1} & B_G \end{array}$$

which extends i to a map f over 1_{B_G}. Then we have shown the following.

PROPOSITION 16. *Let G be compact Lie group acting on a non-singular projective variety V so that the hyperplane section inclusion $i : W \to V$ is a homotopy G-map. Then $\mathrm{tr}(G,V)\big|\deg(V)$.*

Browder and Katz have an example ([**B-K**]) where G acts freely on V and satisfies the hypothesis of Proposition 15, so $|G|\ \big|\ (\deg V)^2$. But $|G|$ does not divide $\deg(V)$. Hence we can conclude that in that example the inclusion of the hyperplane section is not a homotopy G-map.

EXAMPLE: Suppose that $M = S^{2n} \times \cdots \times S^{2n}$ is a product of k spheres each of dimension $2n$. Suppose G acts on M leaving a $2n(k-1)$-dimensional submanifold W invariant. The Poincare dual of W is a class $\alpha \in H^{2n}(M)$ and so $\alpha = a_1\alpha_1 + \cdots + a_k\alpha_k$ where the a_i are integers and α_i generate $H^{2n}(S^{2n})$ of the i-th sphere in the product M. Now $\alpha_i^2 = 0$ for all i and $\alpha_i\alpha_j = \alpha_j\alpha_i$ since the α_i are even dimensional. Hence $\alpha^k = k!(a_1 \dots a_k)\alpha_1 \cup \cdots \cup \alpha_n = k!(a_1 \dots a_k)[\overline{M}]$. Thus $\mathrm{tr}(G, M)$ divides $k!(a_1 \dots a_k)$. More generally, if we have k G–invariant submanifolds W_i (repetitions allowed), each having co-dimension 2n, then their Poincare duals are of the form $\Sigma a_{ij}\alpha_j$. Their intersection number is the permanent of (a_{ij}), and it is divisible by $\mathrm{tr}(G, M)$.

As a particular example, suppose G is an elementary abelian p-group. By ([**G2**], 7.4) we have $\mathrm{tr}(G, M) =$ (the number of points in the smallest orbit). Thus (the number of points in the smallest orbit) divides $k!(a_1 \dots a_k)$.

In the odd dimensional situation, where $M = S^{2n+1} \times \cdots \times S^{2n+1}$ is a product of k odd dimensional spheres each of dimension $2n + 1$, suppose we have a set $\{W_i\}$ of k G-invariant submanifolds of co–dimension $2n + 1$. Then the Poincare dual of each manifold is given by $\beta_i = \Sigma a_{ij}\alpha_j \in H^{2n+1}(M)$ where the a_{ij} are integers and each α_j generates $H^{2n+1}(S^{2n+1})$. Now $\alpha_j \cup \alpha_k = -\alpha_k \cup \alpha_j$. Hence $W_1 \bullet \cdots \bullet W_k = \det(a_{ij})$. So if G is an elementary abelian p-group we see that (the number of points in the smallest orbit) divides $\det(a_{ij})$. In addition, suppose that $k = 2l$ and W is an invariant oriented closed manifold of codimension $2(2n+1)$. Suppose the Poincare dual of W is given by $\beta = \Sigma a_{ij}\alpha_i\alpha_j$. Now $\beta^l = l!\,\mathrm{Pf}(a_{ij})\alpha_1 \dots \alpha_k$. Thus $\mathrm{tr}(G, M)$ divides $l!\,\mathrm{Pf}(a_{ij})$. Here $\mathrm{Pf}(a_{ij})$ is the Pfaffian.

Theorem 7.4 of [**G2**] is greatly generalized by theorem 1.1 of [**B**]. In particular, Browder's result implies that the trace of an action of an elementary abelian p–group on a G–CW complex is equal to the number of points in the smallest orbit.

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