Zeroes of Pullback Vector Fields and Fixed Point Theory for Bodies
Daniel Henry Gottlieb

1. Introduction.

Morse's equation, which is the main tool, is equation (7). It is used to prove several different equations, many of which are described briefly in this introduction and more fully in the body of the paper.

To motivate the concept of the pullback vector field, which is the first item in the title and the first subject of the paper, we consider the following question. Let V be a vector field on $\mathbb{R}^n$. Suppose that M^n is a smooth manifold of dimension n, and suppose that $f : M \to \mathbb{R}^n$ is a smooth map. We ask the question: Is there a vector field V^* on M so that $f_*(V^*(m)) = V(f(m))$ for every $m \in M$? Here $f_* : TM \to T\mathbb{R}^n$ is the differential of f on the tangent bundles.

Such a vector field V^* need not exist. But there is a vector field on M which always exists which we call the pullback vector field f^*V. It is defined in section 4. The index of f^*V equals the index of V^* when it exists. And in in the case where f restricted to ∂M is an immersion and V has no zeroes on the image of $f|\partial M$ we calculate the index in Theorem 5 which gives the following formula.

$$\text{Ind}(f^*V) = \sum_i \omega_i v_i + \Big(\chi(M) - \deg \hat{N}\Big) \tag{1}$$

where V has only isolated zeroes indexed by i and v_i represents the index and ω_i the winding number of the i^{th} zero. The term $\chi(M) - \deg \hat{N}$ is zero for the odd dimensional case, and $\chi(M)$ is the Euler–Poincaré number of M and $\deg \hat{N}$ is the normal degree of the imbedding $f : \partial M \to \mathbb{R}^n$. In fact, $\hat{N}$ represents the Gauss map or normal map. Its degree is called the normal degree or Curvatura Integrala of an immersion. In some sense this equation is a generalization of the global Gauss-Bonnet Theorem.

A local way to calculate $\text{Ind}\, f^*V$ yields the equation

$$\text{Ind}(f^*V) = \sum_i n_i v_i + \sum_j r_j \tag{2}$$

where n_i is the number of regular points in the inverse image of the i^{th} zero and r_j represents the index of the j^{th} "latent" zero of f^*V. It is the existence of latent zeros of f^*V which obstructs the existence of V^*. These two formulas depend only on quantities which can be calculated in $\mathbb{R}^n$. They imply a generalization of the theorem of Haefliger, [8] which states that the normal degree of the boundary of a codimension zero immersion $M^n \to \mathbb{R}^n$ is equal to $\chi(M)$.

In section 7 we use the index of vector fields to study fixed points for compact manifolds in the same dimensional Euclidean space. Suppose that $M^n \subset \mathbb{R}^n$ is a compact body in Euclidean space. If it undergoes a transformation $f : M \to \mathbb{R}^n$, what points remain fixed? The index of a vector field V_f defined on M using f gives us a means to devise a fixed point index in this case. Using this, we find that if $f : M \to M$ has no fixed points on ∂M,

$$\Lambda_f + \Lambda_{-\hat{V}_f, \hat{N}} \hat{N} = \chi(M). \tag{3}$$

Here Λ_f is the Lefschetz number for f and $\Lambda_{-\hat{V},\hat{N}}$ is the coincidence numbers of two maps. $\hat{N}$ is the Gauss map $\partial M \to S^{n-1}$ and $-\hat{V}_f$ is the "Gauss map" for the vector field $-V_f$. Thus we have a topological description of the difference between Λ_f and $\chi(X)$ for any finite complex X. No matter what the thickening of X is, the two Gauss maps defined on the boundary always have the same coincidence number.

We say that $f : M^n \to \mathbb{R}^n$ is *transverse* if the line from m to $f(m)$ is not tangent to ∂M at m. Then we see that transverse maps must have a fixed point if $\chi(M) \not\equiv 0(\mathrm{mod}\, 2)$. Indeed if M is even dimensional, a transverse map has a fixed point if $\chi(M) \neq 0$. These methods imply Dold's generalization of the Brouwer fixed point theorem found in [3]. If $f : D^n \to \mathbb{R}^n$ so that either $x - f(x)$ never points normally inward, or else never points normally outward, where $x \in \partial D^n = S^{n-1}$, then f must have a fixed point.

In section 8 we consider another type of function, the restriction to M of the projection of $\mathbb{R}^n$ to a plane Π^k of dimension k. Then the study of the index of V_f results in an equation

$$\chi(M) = (-1)^{n-k}\chi(\Pi \cap M) + \sum_i \text{ convexity numbers.} \tag{4}$$

The convexity numbers depend on the curvature of ∂M and the position of Π^k and normal lines from ∂M to Π. They maybe related to the convexity numbers defined in a more general context in [1].

For example, if Π^0 is a point in $\mathbb{R}^3$, then $k = 0$ and $n = 3$. Then we have

$$\chi(M) = -\chi\left(\Pi^0 \cap M\right) + \sum_i c_i \tag{5}$$

where c_i is calculated as follows in the generic case. The point Π^0 lies on a set of lines normal to ∂M so that as one moves along the line from ∂M to Π, one enters into the body M. These lines are indexed by i. Now let us imagine that Π moves along some normal ℓ_i. When $\Pi = x_i \in \partial M$ we let $c_i = +1$. As Π moves along the normal and crosses a point which is the center of curvature for one of the principal curves of curvature for x_i, then the c_i should change sign. Note that $\chi(\Pi \cap M) = 0$ or 1 according to whether Π is outside or inside M.

Thus for example, if Π^0 is inside a diffeomorphic 3–ball, the sum total of principal radii of curvature it crosses as it moves toward ∂B along all exiting normals must be even. If Π^0 were outside, it must be odd.

Another example when $k = 1$ and $n = 2$. Then Π is a line in the plane. Then (4) reduces to

$$\chi(M) = -\chi(\Pi \cap M) + \Sigma c_i. \tag{6}$$

In this case note that $\chi(\Pi \cap M)$ is the number of components of the line in M. Also c_i is computed as follows. Let i index those lines in the plane which are normal to Π and to ∂M, and so that traveling from ∂M towards Π, one initially enters into M. Then if ∂M curves towards Π at χ_i, let $c_i = +1$ and $c_i = -1$ if ∂M curves away.

In the case of the plane ($k = 2$) in $\mathbb{R}^3$, the i indexes lines normal to Π and ∂M which enter M as one moves from ∂M to Π. Then the c_i should equal the sign of the curvature at x_i.

A different relationship involving Euler-Poincaré numbers and curvature is given in theorem 15. We give a very special consequence of it here. Suppose a line intersects a solid torus in two segments. Then there are four points on the boundary torus with negative curvature whose tangent planes all contain the line.

Our methods are based on the following index formula. Let M denote a compact manifold, with or without boundary ∂M. Let V be a tangent vector field on M. Suppose that V has no zeroes on ∂M. There is an integer associated to V called the index of V, denoted $\mathrm{Ind}(V)$ here. This index satisfies the following equation, [11], [12], [9], [3], [4],

$$\mathrm{Ind}(V) + \mathrm{Ind}(\partial_- V) = \chi(M). \tag{7}$$

Here $\chi(M)$ denotes the Euler–Poincaré number of M, and $\partial_- V$ is a vector field defined on part of ∂M as follows. Let $\partial_- M$ be the subset of ∂M containing all m so that $V(m)$ is pointing into the manifold. Then $\partial_- V$ is the vector field given by first restricting V to ∂M, and then projecting the vectors to ∂M so as to get a vector field tangent to ∂M, and finally restricting ∂V to $\partial_- M$.

This equation could almost serve as a definition of the index of a vector field. All the properties of the index should be derivable from it. Yet, for the past sixty years, it has largely been ignored. What we do here is look at various situations in topology where a vector field can be defined, and study the different interpretations of the least familiar term, $\mathrm{Ind}(\partial_- V)$. In the next section, we will show that $\mathrm{Ind}(\partial_- V)$ equals a coincidence number under some circumstances.

2. Coincidence numbers.

We want to show that $\mathrm{Ind}(\partial_- V)$ equals a certain coincidence number. In this section we study the local situation.

Let $\alpha : D^{n-1} \to \mathbb{R}^n$ be a smooth imbedding of an $n-1$ dimensional disk into n–dimensional Euclidean space. Suppose that V is a vector field on $\mathbb{R}^n$ which is not zero on the image of D in $\mathbb{R}^n$. That is,

$$V(\alpha(m)) \neq \vec{0} \in T_{\alpha(m)}(\mathbb{R}^n) \text{ where } m \in D.$$

Let N be a vector field on $\alpha(D)$ of unit length normal to $\alpha(D)$. It will be necessary to choose orientations of D and $\mathbb{R}^n$ in order to use coincidence numbers. But since the index of a vector field does not depend upon the choice of orientations, it is necessary to be careful. We relate the orientations of D^{n-1} and $\mathbb{R}^n$ as follows.

Choose a basis of the tangent space at m, denoted $T_m(D)$. Call the basis $b_1, \ldots, b_{n-1}$ and give it the ordering which agrees with the orientation of D. Then $\{\alpha_*(b_1), \ldots, \alpha_*(b_{n-1}), N(\alpha(m))\}$ is a basis for $T_{\alpha(m)}(\mathbb{R}^n)$. Then we let the ordering $\{\alpha_*(b_1), \ldots, \alpha_*(b_{n-1}), N(\alpha(m))\}$ represent the orientation of $T_{\alpha(m)}(\mathbb{R}^n)$.

Now, using N, we want to project $V|\alpha(D)$ down onto a vector field V' which is tangent to $\alpha(D)$. We define V' by

$$V'(\alpha(m)) = V(\alpha(m)) - [V(\alpha(m)) \bullet N(\alpha(m))]N(\alpha(m)). \tag{8}$$

Or more abbreviated, $V' = V - (V \bullet N)N$, where $\bullet$ represents the dot product. Next we pullback V' to a vector field V^* using α. We define V^* on D by $V^*(m) = \alpha_*^{-1}(V'(\alpha(m)))$. This is well defined since α is an imbedding.

The vector field N gives rise to a Gauss map or normal map $\hat{N} : D \to S^{n-1}$. This is defined by parallel translating the normal vector $N(\alpha(m))$ to a vector based at the origin of $\mathbb{R}^n$. Then the tip of this vector lies on the unit sphere. We introduce the concept of the *Gauss map* $\hat{V}$ for any non zero vector field. Then $\hat{V} : D \to S^{n-1}$ is given by making V of unit length and parallel translating the vector to the origin, and looking at its end point. Thus

$$m \mapsto V(\alpha(m)) \mapsto \frac{V(\alpha(m))}{\|V(\alpha(m))\|} \mapsto \hat{V}(m)$$

where $\hat{V}(m)$ a parallel vector of unit length based at the origin.

Now V^* is related to $\hat{V}$ and $\hat{N}$ as follows. The point $m_0 \in D$ is a coincidence of $\hat{V}$ and $\hat{N}$, that is $\hat{N}(m_0) = \hat{V}(m_0)$, if and only if $N(m_0) \bullet V(m_0) > 0$ and $V^*(m_0) = \vec{0} \in T_{m_0}(D)$. Now this is obvious. What is not so obvious, and needs a careful argument, is that for isolated coincidence points m_0, the index of V^* at m_0 is equal to the coincidence number, $\mathrm{Coinc}(\hat{N}, \hat{V})$.

We follow the excellent, and perhaps unique, account of coincidence [13], see chapter 6. Suppose we have two *oriented* manifolds M^n and N^n. Suppose that $f, g : M \to N$ are two maps with an isolated coincidence point m_0. Then the coincidence number of f and g at m_0, denoted $\mathrm{Coinc}(f, g; m_0)$, is defined as follows. Choose two homeomorphisms of the unit disk $D \subset \mathbb{R}^n$, denote them by $h : D \to M$ and $k : D \to N$. We choose h and k so that $h(0) = m_0$ and $k(0) = f(m_0) = g(m_0)$. Also we want the image of $h(D) \subset M$ under both f and g to be contained in $k(D)$, so that $k^{-1}fh$ and $k^{-1}gh$ are defined. Also we choose h so that $h(D)$ contains only one coincidence point, m_0. And last, but very important, we choose h and k so that $k^{-1}h : D \to D$ preserves orientation. Now define

$$\varphi : S^{n-1} = \partial D \xrightarrow{h} M \xrightarrow{(f,g)} N \times N \xrightarrow{k^{-1} \times k^{-1}} D \times D \xrightarrow{F} D - \vec{0} \tag{9}$$

where $F(x, y) = \frac{1}{2}(y - x)$. In other words,

$$\varphi(m) = \frac{1}{2}\left(k^{-1}gh(m) - k^{-1}fh(m)\right) \in D - \vec{0}. \tag{10}$$

Then the degree of φ is defined to be $\mathrm{Coinc}(f, g; m_0)$. If $M_1 = M_2$ and the identity is orientation preserving, then $\mathrm{Coinc}(f, 1; m_0)$ is the usual fixed point index. This definition is independent of the choices made. Also the definition is stable under homotopies of f and g such that no coincidence point of f_t and g_t, for any t, lies on the boundary of $h(D)$. However, if the orientation of either M or N is reversed, then the sign of the coincidence number changes. And if the roles of f and g are reversed, then

$$\mathrm{Coinc}(f, g) = (-1)^n \,\mathrm{Coinc}(g, f) \text{ where } n = \dim M. \tag{11}$$

Lemma 1. *If m_0 is an isolated coincidence of $\hat{N}$ and $\hat{V}$, then* $\operatorname{Ind}(V^*) = \operatorname{Coinc}\left(\hat{N}, \hat{V}; m_0\right)$.

Proof. Without loss of generality, we can assume that the imbedding $\alpha : D^{n-1} \to \mathbb{R}^n$ takes the unit disk D in $\mathbb{R}^{n-1}$ into $\mathbb{R}^n$ so that the coincidence point $m_0 = \vec{0}$. Assume also that $N(0) = (0, 0, \ldots, 0, 1)$, so that $\hat{V}(0) = \hat{N}(0)$ is the "north pole" of $S^{n-1} \subset \mathbb{R}^n$. We can assume also that $\alpha(D)$ is small enough so that the images of $\hat{V}$ and $\hat{N}$ lie in a small neighborhood of the north pole in S^{n-1}. Let $\pi : S^{n-1} \to D^{n-1}$ be the projection. It gives a homeomorphism between the northern hemisphere and the equitorial disk. In our calculation of $\operatorname{Coinc}(\hat{N}, \hat{V})$, the role of h will be played by the identity and the role of k by π^{-1}. The choice of orientations is given by choosing an orientation of $\mathbb{R}^n$. This induces orientations on the northern hemisphere and on D^{n-1} and π is orientation preserving. The outward pointing normal to S^{n-1} induces the orientation on S^{n-1}.

Let $\alpha_t : D \to \mathbb{R}^n$ be an isotopy of embeddings so that $\alpha_t(0)$ is fixed. Then we have normal fields $N_t(m)$ for each $t \in I$, and each N_t is defined on $\alpha_t(D)$. We choose the isotopy α_t so that $N_t(0) = N(0)$. More precisely, $N_t(m) \in T_{\alpha_t(m)}(\mathbb{R}^n)$ so that $\|N_t(m)\| = 1$ and $N_t(m)$ is orthogonal to $(\alpha_t)_*(T_m(D))$. Finally we assume that α_1 = translation of D, (so that 0 goes to $N(0)$ in the $(n-1)$–dimensional hyperplane passing through $N(0)$ and orthogonal to $N(0)$). So α_t can be thought of as a deformation of $\alpha(D)$ which flattens it out into the tangent hyperplane at $\alpha(0)$. We assume that α_1 translates D to the unit disk tangent to the north pole.

Now we define a family of vector fields V_t, each one defined on the corresponding subspace $\alpha_t(D)$ of $\mathbb{R}^n$, by letting

$$V_t(m) = [V(\alpha(m)) \bullet N(m)] N_t(m) + (\alpha_t)_* (V^*(m)) \qquad \in T_{\alpha(m)}(\mathbb{R}^n). \tag{12}$$

Then we have $(\alpha_t)_*(V^*) = V_t$ for every t.

Now we consider $\hat{N}_t$ and $\hat{V}_t$. For any t, the maps $\hat{N}_t$ and $\hat{V}_t$ have only one coincidence point, at 0, since $V_t = V^*$ has only one zero, at 0. That being the case,

$$\operatorname{Coinc}(\hat{N}, \hat{V}) = \operatorname{Coinc}(\hat{N}_t, \hat{V}_t) = \operatorname{Coinc}(\hat{N}_1, \hat{V}_1). \tag{13}$$

Recall that $\hat{V}_1(m)$ is parallel to $N(\vec{0})$. Thus

$$V_1(m) = [V(\alpha(m)) \bullet N(m)] N_1(m) + \alpha_{1_*}(V^*(m)). \tag{14}$$

In addition we have, where v_0 denotes v_m translated to the origin,

$$\begin{aligned} &\hat{N}_1(m) = \text{ north pole; } \hat{V}_1(m) = \\ &\frac{\left([V(\alpha(m)) \bullet N(m)] N(\vec{0}) + (\alpha_{1_*}(V^*(m)))_0\right)}{\left((V(\alpha(m)) \bullet N(m))^2 + \|\alpha_{1_*}(V^*(m))\|^2\right)^{\frac{1}{2}}}. \end{aligned} \tag{15}$$

Recall that π :*Northern hemisphere* $\to D$ was the projection and $\pi \circ \alpha_1 =$ identity. Then $\pi \circ \hat{N}_1 = \vec{0}$ and $\pi \circ \hat{V}_1(m) = k(m)\left(\alpha_{1^*}(V^*(m))\right)_0 = k(m)(V^*(m))_0$ where $k(m) = \left[(V(\alpha(m)) \bullet N(m))^2 + \|\alpha_{1_*}(V^*(m))\|^2\right]^{-\frac{1}{2}} > 0$. Now by (10), the coincidence number $\mathrm{Coinc}(\hat{N}_1, \hat{V}_1)$ is the degree of $\varphi : \partial D \to \mathbb{R}^{n-1} - \vec{0}$, where

$$\varphi(m) = \frac{1}{2}\left(\pi \circ \hat{V}_1(m) - \pi \circ \hat{N}_1(m)\right) = \frac{1}{2}k(m)\left(\alpha_{1_*}(V^*(m))\right)_0 \in \mathbb{R}^{n-1} \tag{16}$$

On the other hand, the index of a vector field V is given by the degree of the map

$$\psi : \partial D \to \mathbb{R}^{n-1} - 0 \text{ where } \psi(m) - (V(m))_0. \tag{17}$$

Thus we have shown that

$$\mathrm{Coinc}(\hat{N}, \hat{V}) = \mathrm{Coinc}(\hat{N}_1, \hat{V}_1) = \deg \varphi = \mathrm{Ind}\left(\frac{k(m)}{2}V^*\right). \tag{18}$$

Now $\frac{k}{2} : D \to \mathbb{R}_+$ is homotopic to the constant 1, and since this homotopy does not introduce new zeroes, we see that $\mathrm{Ind}\left(\frac{k(m)}{2}V^*\right) = \mathrm{Ind}(V^*)$. This proves the lemma.

3. Codimension one immersions.

In this section we consider the following situation. Let C^{n-1} be a smooth closed oriented manifold. Suppose that $\alpha : C^{n-1} \to \mathbb{R}^n$ is a smooth immersion. Let V be a vector field on $\mathbb{R}^n$ which is not zero on the image $\alpha(C^{n-1}) \subset \mathbb{R}^n$. We choose an orientation of $\mathbb{R}^n$. Now for each point $m \in C$, we choose a vector $N(m) \in T_{\alpha(m)}(\mathbb{R}^n)$ such that $N(m)$ has unit length and $N(m)$ is orthogonal to the tangent space of $\alpha(C)$ at $\alpha(m)$. Now N induces an orientation on C exactly as in the last section. Note that N is not a vector field on $\alpha(C)$ since α need not be injective. Now for each $m \in C$, we define a vector $V'(m) \in T_{\alpha(m)}(\mathbb{R}^n)$ by

$$V'(m) = -[V(\alpha(m)) \bullet N(m)]N(m) + V(\alpha(m)).$$

Thus $V'(m) \bullet N(m) = 0$, so $V'(m)$ is tangent to $\alpha(C)$ at $\alpha(m)$. Again V' does not define a vector field on $\alpha(C)$. We define a vector field V^* on C by the equation

$$\alpha_*(V^*(m)) = V'(m).$$

Now V^* is well defined since α is an immersion, so α_* is injective, and $V'(m)$ is in the image of α_*.

We denote by V^*_+, the vector field V^* restricted to the open subspace of C such that $V(\alpha(m)) \bullet N(m) < 0$. Note that every zero of V^* is either in V^*_+ or V^*_- since $V(\alpha(m))$ is never zero. Thus

$$\mathrm{Ind}\, V^*_+ + \mathrm{Ind}\, V^*_- = \mathrm{Ind}(V^*) = \chi(C). \tag{20}$$

We recall here some more coincidence theory, suppose M_1 and M_2 are both closed, oriented manifolds of dimension n. Suppose f and g are maps from M_1 to M_2. Suppose that f and g have a finite number of coincidence points, M_i. Then we introduce the Lefschetz trace

$$\Lambda_{f,g} = \sum_j (-1)^j tr(g^! f^*)_j. \tag{21}$$

Here $f^* : H^j(M_2) \to H^j(M_1)$ is the induced map in cohomology and $g^! : H^j(M_1) \to H^j(M_2)$ is the umkehr map, defined by $g^! = D^{-1} g_* D$ where $D : H^j(M) \to H_{n-j}(M)$ is the Poincaré duality map. Also $tr(f^* g^!)_j$ denotes the trace of $f^* g^! : H^j(M_1) \to H^j(M_1)$.

Note that $g^!$ reverses sign when the orientation of either M_1 or M_2 is reversed. If $M_1 = M_2$ and the orientations are chosen so that the identity 1 preserves orientation, then $\Lambda_{f,1} = \Lambda_f$ where Λ_f is the usual Lefschetz number. Now the main result of coincidence theory is

$$\Lambda_{f,g} = \sum_i \mathrm{Coinc}(f, g; m_i). \tag{22}$$

This generalizes the result for fixed points.

Lemma 2. *Suppose that C^{n-1} is immersed into $\mathbb{R}^n$ as above, so that V^* has only a finite number of zeroes. Then*

$$\mathrm{Ind}\, V_+^* = \Lambda_{\hat{N},\hat{V}} = \deg \hat{V} + (-1)^{n-1} \deg \hat{N}, \text{ if } n-1>0.$$

Proof. Recall that $\hat{N}$ and $\hat{V}$ are the Gauss maps from $C^{n-1} \to S^{n-1}$ given by taking the vector associated to m, making it of unit length, and translating it to the origin. Now

$$\Lambda_{\hat{N},\hat{V}} = \sum_i \mathrm{Coinc}\left(\hat{N}, \hat{V}; m_i\right) = \sum_i \mathrm{Ind}\left(V_+^*; m_i\right) = \mathrm{Ind}\, V_+^*$$

where m_i are the isolated coincidence points of $\hat{N}$ and $\hat{V}$, which correspond to those zeroes of V^* that are in V_+^*. This follows by applying equation (22) and Lemma 1. From equation (21) we see that

$$\begin{aligned} \Lambda_{\hat{N},\hat{V}} &= tr\left(\hat{V}^! \hat{N}^*\right)_0 + (-1)^{n-1} tr\left(\hat{V}^! \hat{N}^*\right)_{n-1} \\ &= \deg \hat{V} + (-1)^{n-1} \deg \hat{N}. \end{aligned}$$

Lemma 3. $\mathrm{Ind}\, V_-^* = \Lambda_{-\hat{V},\hat{N}} = -\deg \hat{V} + \deg \hat{N}$, *if* $n-1>0$.

Proof. We apply lemma 2 to the vector field $-V$ and obtain

$$\mathrm{Ind}(-V)_+^* = \Lambda_{\hat{N},-\hat{V}} = \deg(-\hat{V}) + (-1)^{n-1} \deg \hat{N}.$$

Now $(-V)^*_+ = -(V^*_-)$, so $\operatorname{Ind} V^*_- = (-1)^{n-1} \operatorname{Ind}(-(V^*_-)) = (-1)^{n-1} \operatorname{Ind} V^*_+$. Also $\Lambda_{\hat{N},-\hat{V}} = (-1)^{n-1}\Lambda_{-\hat{V},\hat{N}}$. And $\deg(-\hat{V}) = (-1)^n \deg \hat{V}$ since $-\hat{V}$ is just $\hat{V}$ composed with the antipodal map. Substitution into the equation gives the desired result.

Now adding the equations of Lemmas 2 and 3 and using equation (20) we obtain

$$\chi(C) = \left(1 + (-1)^{n-1}\right) \deg \hat{N}.$$

When C is odd dimensional we obtain nothing, but for even dimensional C we see that

$$\deg \hat{N} = \frac{1}{2}\chi(C) \text{ for } \dim C > 0 \text{ even.} \tag{23}$$

This was originally discovered by H. Hopf. Hopf called $\deg \hat{N}$ the *curvature Integrala*. Another name is *normal degree*. Milnor [10] and Bredon–Koscinski [2] showed that for odd dimensional connected C the normal degree can take on any value if and only if C is parallelizable. If C is connected and not parallelizable, then the normal degree may take on any odd value, but no even values.

We can give a nice formula for $\deg \hat{V}$. To do that we need to introduce the concept of the *winding number* of $\alpha : C^{n-1} \to \mathbb{R}^n$ about $p \in \mathbb{R}^n - \alpha(C)$. This is a generalization of the usual notion for closed paths in the plane.

Definition. Choose orientations of C^{n-1} and $\mathbb{R}^n$. Then the *winding number* $w(\alpha, p)$ is the degree of the map $C^{n-1} \to \mathbb{R}^n - p \approx S^{n-1}$.

In the case where α is an immersion, we can give a very geometric way to calculate the winding number. The choice of orientations will give rise to the "vector field" $N(m)$ of normals at $\alpha(m)$. Now consider a path from p out towards infinity which crosses $\alpha(C)$ transversally. For each crossing, assign a $+1$ or a -1 according to whether the path crosses into the "side" of $\alpha(C)$ that $N(m)$ points to or not. The sum of these ±1's is the winding number.

Now suppose that $\alpha : C^{n-1} \to \mathbb{R}^n$ is an immersion as before and suppose that the vector field V on $\mathbb{R}^n$ has isolated zeroes p_i. Let $\omega_i = W(\alpha, p_i)$, the winding number about the i^{th} zero. Let $v_i = \operatorname{Ind}(V, p_i)$, the index of V at the i^{th} zero. Then

Lemma 4.

$$\deg \hat{V} = \sum_i \omega_i v_i.$$

Proof. The fact that C^{n-1} immerses in $\mathbb{R}^n$ implies that it bounds a Π–manifold M. That is, M is parallelizable and $\partial M = C$. So we extend the immersion α to a map $f : M \to \mathbb{R}^n$. We can always do this since $\mathbb{R}^n$ is contractible, and we can do it so that f is smooth and so that the zeroes, p_i, of V are regular points. We think of the vector field V on $\mathbb{R}^n$ as a map $\widetilde{V} : \mathbb{R}^n \to \mathbb{R}^n$ given by $\widetilde{V}(p) = (V(p))_0$. Now consider $\widetilde{V} \circ f : M^n \to \mathbb{R}^n$. Restricted to ∂M, the map is:

$$\widetilde{V} \circ \alpha : \partial M \to \mathbb{R}^n - 0 \subset \mathbb{R}^n.$$

Now $\widetilde{V}$ can be homotopied to $\widetilde{V}_1$ by $\widetilde{V}_t(p) = k_t(p)V(p)$ where $k_t(p) > 0$ is chosen so that $\|\widetilde{V}_1(p)\| \leq 1$ for all $p \in \mathbb{R}^n$ and $\|\widetilde{V}_1(p)\| = 1$ for $p \in \alpha(\partial M)$. Then $\widetilde{V}_1 \circ \alpha = \hat{V}$ and $\widetilde{V}_t \circ \alpha : \partial M \to \mathbb{R}^n - 0$. So $\widetilde{V}_1 \circ f$ can be thought of as a map from $M \to D$ which takes ∂M into ∂D by a map of degree equal to $\deg \hat{V}$. This degree can also be calculated locally by looking at the inverse image of a small neighborhood of $0 \in \mathbb{R}^n$ under $\widetilde{V}_1 \circ f$. The inverse image at 0 of $\widetilde{V}_1$ is just the zeroes of V, namely the p_i. And the inverse images of a p_i are points $m_{ij} \in M$. Now each m_{ij} has a little neighborhood that maps diffeomorphically onto a small neighborhood of p_i. Thus locally, the map $\widetilde{V}_1 \circ f$ restricted to a neighborhood of m_{ij} has degree $\pm v_i$ depending on whether f preserves or reverses the orientation of the neighborhood. Then

$$\deg \hat{V} = \sum_{i,j} \pm v_i = \sum_i \omega_i v_i.$$

Remark. For the results of this section there is no stipulation that the manifold C is connected. However, lemmas 2 and 3 are not true for C of dimension zero since $H_0(S^0) = \mathbb{Z}$.

4. Codimension zero maps.

Let $f : M^n \to \mathbb{R}^n$ be a smooth map from a compact Riemannian manifold M into $\mathbb{R}^n$. Let V be a vector field on $\mathbb{R}^n$. Let $\langle\ ,\ \rangle_R$ the Riemannian metric on $\mathbb{R}^n$.

Definition. the *pullback vector field* f^*V on M is defined by

$$\langle f^*V(m), v_m \rangle_M = \langle V(f(m)), f_*(v_m) \rangle_R.$$

This f^*V can be thought of as the dual to the pullback of a 1–form. Our objective is to prove the following precise statement of equation (1).

Theorem 5. *Suppose that f is a smooth map which restricts to an ambient immersion α on ∂M. We assume that ∂M is orientable and $n > 1$, and no zeroes of V are in the image of ∂M. Then*

$$\operatorname{Ind}(f^*V) = \sum_i \omega_i v_i + \chi(M) - \deg \hat{N}.$$

The condition that α is an ambiant immersion on ∂M means that the singular set of f is disjoint from ∂M. Equivalently, f is an immersion on a neighborhood of ∂M. Here $N(m)$ is defined as before as the unit normal to $\alpha(\partial M)$ at $\alpha(m)$, pointing in the outward direction. Then a choice of an orientation on $\mathbb{R}^n$ induces, by means of $N(m)$, an orientation of each component of ∂M. Note that this orientation of ∂M may not be consistent with any orientation of M, indeed M may not be orientable. But ∂M must be. With this convention, we can apply the results of the last section.

Now since $\alpha_* : T_m(M) \to T_{\alpha(m)}(\mathbb{R}^n)$ is an isomorphism, we may define vector fields $\alpha_*^{-1}N$ and $\alpha_*^{-1}V$ on $T(M)|\partial M$. We also have a vector field f^*N on ∂M.

Lemma 6. *The vector field f^*N is an outward pointing normal vector field on ∂M. The vector field $\alpha_*^{-1}N$ points outwards.*

Proof. Let v_m be any vector tangent to ∂M at m. Then

$$\langle f^*N(m), v_m\rangle_M = \langle N(m), \alpha_*(v_m)\rangle_R = 0.$$

So f^*N is a normal vector field. Now

$$\langle f^*N(m), \alpha_*^{-1}N(m)\rangle_M = \langle N(m), \alpha_*\left(\alpha_*^{-1}N(m)\right)\rangle = 1 > 0.$$

We learn from this that f^*N and $\alpha_*^{-1}N$ both point out of the same side. Since N points outwards locally in $\mathbb{R}^n$, $\alpha_*^{-1}N$ must point outwards also in M. Given any outward pointing vector field on ∂M, we may project any vector field on ∂M down to a vector field tangent to ∂M. Recall the definition of V^* as in §3. Thus $V^*(m) = \alpha_*^{-1}V(\alpha(m)) - k\alpha_*^{-1}(N(m))$ for some number k, chosen so that $V^*(m)$ is tangent to ∂M. Thus we say that V^* is the projection of $\alpha_*^{-1}V$ onto ∂M by means of the outward pointing vector field $\alpha_*^{-1}N$. Similarly, the notation $\partial(\alpha_*^{-1}V)$ represents the projection of $\alpha_*^{-1}V$ onto ∂M by means of the outward pointing normal vector field at ∂M. Recall that is f^*N. Finally ∂f^*V is the projection of f^*V onto ∂M by means of f^*N.

Lemma 7. $\operatorname{Ind} V_-^* = \operatorname{Ind}\partial_-(\alpha_*^{-1}V) = \operatorname{Ind}\partial_-(f^*V)$.

Proof. Here the subscript minus denotes the restrictions of the vector fields to the portion of the boundary where the original field pointed inwards. Now the vector fields $N_t = (1-t)(\alpha_*^{-1}N) + t(f^*N)$ always point outward for $t \in I$, since $\langle N_t, f^*N\rangle > 0$ and f^*N is an outward pointing normal. The homotopy gives rise to a homotopy V_t^* of vector fields tangent to ∂M induced by projecting $\alpha_*^{-1}V$ down to ∂M by means of N_t. If $V^*(m) = V_0^*(m)$ were tangent to ∂M, it would remain constant for all t. Thus $(V_t^*)_-$ is a vector field on the same open subspace of ∂M for all t, and no zeroes of V_t^* will occur on the frontier of this inward pointing region. Hence $\operatorname{Ind}(V_t^*)_-$ is a constant and we have the first equality from $\operatorname{Ind} V_-^* = \operatorname{Ind}(V_t^*)_- = \operatorname{Ind}\partial_-(\alpha_*^{-1}V)$. Next we consider the vector fields

$$V_t = t(\alpha_*^{-1}V) + (1-t)(f^*V).$$

This homotopy on the boundary can be extended to a homotopy $\overline{V}_t$ of f^*V over M. Now on the boundary V_t is never zero. This can be seen since

$$\langle f^*V(m), \alpha_*^{-1}V(\alpha(m))\rangle_M = \langle V(\alpha(m)), V(\alpha(m))\rangle > 0.$$

Thus $\langle V_t, f^*V\rangle_M > 0$. Hence the homotopy $\overline{V}_t$ never has a zero on ∂M. Hence $\operatorname{Ind} f^*V = \operatorname{Ind}\overline{V}_t = \operatorname{Ind}\overline{V}_1$. Now by equation (7),

$$\operatorname{Ind}(\overline{V}_t) + \operatorname{Ind}(\partial_-\overline{V}_t) = \chi(M).$$

So $\operatorname{Ind}(\partial_-(f^*V)) = \operatorname{Ind}(\partial_-(\alpha_*^{-1}V))$, proving the second inequality.

Proof of theorem 5.

$$\begin{aligned}\operatorname{Ind} f^*V &= \chi(M) - \operatorname{Ind}(\partial_- f^*V)\\ &= \chi(M) - \operatorname{Ind}(V_-^*)\\ &= \chi(M) - (-\deg \hat{V} + \deg \hat{N})\\ &= \chi(M) + \Sigma\omega_i v_i - \deg \hat{N}.\end{aligned}$$

Remark. Note that there is no choice of orientations in theorem 5. The outward pointing normals are all that are needed to calculate ω_i or $\deg \hat{N}$. In the proof, careful attention was paid to how the orientations were chosen. If the orientation of $\mathbb{R}^n$ is reversed, our conventions assure us that the orientations of ∂M and S^{n-1} are reversed. If we changed our convention so that $\{N, b_1, \ldots, b_n\}$ determines the orientation instead of $\{b_1, \ldots, b_n, N\}$, the consistent use of this would change both the orientations of ∂M and S^{n-1} and so we still have the correct equation. The sign changes always cancels out.

We say that a vector field V^* on M *lifts* a vector field V on $\mathbb{R}^n$ if $f_*(V^*(m)) = V(m)$ for all $m \in M$.

Proposition 8. *If V^* lifts V, then* $\operatorname{Ind} V^* = \operatorname{Ind} f^*V$.

Proof. Consider the homotopy of vector fields

$$V_t = tV^* + (1-t)f^*V \text{ for } t \in I.$$

Note that if m is not a zero of V^* and f^*V, then m is not a zero of V_t. Since both V^* and f^*V are never zero on ∂M, we see that V_t is never zero on ∂M. Hence $\operatorname{Ind} V_0 = \operatorname{Ind} V_1$.

Corollary 9. *If $f : M^n \to \mathbb{R}^n$ has a lifted vector field V^*, where V has no zeroes, then* $\deg \hat{N} = \chi(M)$.

Proof. $0 = \operatorname{Ind} V^* = \operatorname{Ind} f^*V = \chi(M) - \deg \hat{N}$.

If $f : M^n \to \mathbb{R}^n$ were an immersion, every vector field on $\mathbb{R}^n$ has a lifting V^*. Hence we have the theorem of Haefliger [8].

Corollary 10. *If $f : M^n \to \mathbb{R}^n$ is an immersion, then* $\deg \hat{N} = \chi(M)$.

5. Lifting vector fields.

In this section we shall draw some consequences of Theorem 5 and Proposition 8. First we obtain a formula for $\operatorname{Ind}(f^*V)$ in terms of local conditions. Then we compare it with theorem 5, which is a global formula.

Consider as usual a smooth map $f : M^n \to \mathbb{R}^n$. Suppose that V is a vector field on $\mathbb{R}^n$ with isolated zeroes at x_i. For each zero x_i, let n_i stand for the number of noncritical points in $f^{-1}(x_i)$. Now consider the zeroes of f^*V. They fall into two classes. There are the *regular* zeroes, that is, those zeroes which are non singular points of f; and the *latent* zeroes, those zeroes which occur at the singular points of f. The image of a regular zero is a zero of V, but the image of a latent zero need not be a zero of V. Now if m_j is a latent zero, we will denote its local index by r_j. If m is a regular zero, then the local index of

m equals v_i, the local index of $x_i = f(m)$. This is seen since f restricted to a small neighborhood of m is a diffeomorphism, hence locally by Proposition 8, $\operatorname{Ind} f^*V = \operatorname{Ind} V^* = \operatorname{Ind} V$. The second equality follows from (7). It is the fact that index is invariant under diffeomorphism.

Proposition 11. $\operatorname{Ind} f^*V = \Sigma n_i v_i + \Sigma r_j$ *when the* n_i *are finite. Then in the case where the singular set* Σ *does not intersect* ∂M*, we combine Proposition 11 and Theorem 5 and Proposition 8, if* V^* *exists, to get the equation*

$$\operatorname{Ind} V^* = \operatorname{Ind} f^*V = \Sigma \omega_i v_i + \chi(M) - \deg \hat{N} = \Sigma n_i v_i + \Sigma r_j \tag{24}$$

Now we draw some consequences of equation (24). We introduce the following notation. Given $f : M^n \to \mathbb{R}^n$ with a vector field V defined on $\mathbb{R}^n$, we let $Z \subset \mathbb{R}^n$ denote the set of zeroes of V, and Σ denote the singular points of f and $\Delta = f(\Sigma)$. We call Δ the *discriminant* of f. In this terminology, the conditions for equation (24) to hold are $\partial M \cap \Sigma = \phi$ and $f(\partial M) \cap Z = \phi$ and $f^{-1}(Z) - \Sigma$ is finite.

Then applying (24) we see

(25) $\operatorname{Ind} f^*V = \chi(M) - \deg \hat{N} = \Sigma r_j$ if $Z = \phi$.

(26) $\operatorname{Ind} f^*V = 0 = \Sigma r_j$ if $Z = \phi$ and n odd.

(27) $\operatorname{Ind} V^* = \Sigma \omega_i v_i + \chi(M) - \deg \hat{N} = \Sigma n_i v_i$ if $Z \cap \Delta = \phi$.

(28) $\operatorname{Ind} V^* = \Sigma \omega_i v_i = \Sigma n_i v_i$ if $Z \cap \Delta = \phi$ and n odd.

The condition that the vector fields involved have isolated singularities is not meant seriously. In fact, we can define the local index of isolated connected components of Z, by using equation (7). So for each connected component Z_i of Z, we define the local index v_i. Since $Z \cap f(\partial M) = \phi$, we see that ω_i is constant at every point of Z_i. So the first two equations of (24) are true in this sense.

The last equation of (24) makes sense if we add the conditions that $f(\partial M) \cap \Delta = \phi$ and that $Z_i \subset \Delta$ if Z_i intersects Δ and $f^{-1}(Z)$ consists of a finite number of path components. Then the r_j are the indices of those components of the zeroes of f^*V, or of V^*, which are contained in Σ. And v_i is the index of the sets Z_i in $\mathbb{R}^n$. The n_i are constant for each point in Z_i since $f(\partial M) \cap \Delta = \phi$.

Remark. The reader may interpret ω_i as a local degree for the map f. This is not quite accurate. We will give an example which should eliminate some misconceptions. Let M be the Mobius band. Hence M is not orientable! Let $f : M \to \mathbb{R}^2$ be a map so that Σ is the middle circle of the Mobius band. Suppose that $\Delta = f(\Sigma)$ is the unit circles in $\mathbb{R}^2$. Suppose that the rest of M is mapped outside of the unit circle and that $\partial M = S^1$ is mapped with winding number 2 about the origin. Now let V be the vector field given by an instantaneous rotation of $\mathbb{R}^2$ about the origin. Then the origin is the only zero of V and it has index $v = 1$. Note that this zero is not in the image of f! Now V lifts to a vector field V^* which has no zeroes. Hence applying theorem 5

$$0 = \operatorname{Ind} V^* = 2 \times 1 + 0 - 2$$

which verifies that the equation is true in this situation.

6. The Euler–Poincaré number of $f^{-}(f(\Sigma))$.

Here we obtain a formula for $\chi(f^{-1}(f(\Sigma))$ where $f : M^n \to \mathbb{R}^n$ is a map satisfying mild conditions by lifting vector fields. We define a vector field V on $\mathbb{R}^n$ which can be lifted, and using (24) we determine the index of the lifting V^* in two ways. Combining them yields the formula.

Theorem 12. *Let $f : M^n \to \mathbb{R}^n$ be a smooth map such that $f(\Sigma) \cap f(\partial M) = \phi$. Suppose there exists a compact submanifold $N^n \subset \mathbb{R}^n$ so that $f(\Sigma)$ is a deformation retract of N and $f^{-1}(f(\Sigma))$ is a deformation retract of $f^{-1}(N)$.*

Then

$$\chi(f^{-1}(f(\Sigma))) - \sum_i \omega_i \chi(\Sigma_i) + (-1)^n \sum_j (\omega_j - n_j)\chi(D_j) + \chi(M) - \deg \hat{N}. \tag{29}$$

Here Σ_i are the connected components of $f(\Sigma)$. The D_j are the bounded components of $\mathbb{R}^n - f(\Sigma)$. For each D_j we choose a point $x_j \in D_j$. Then n_j is the cardinality of $f^{-1}(x_j)$ and ω_i is the winding number about Σ_i.

Proof. We construct a vector field V on $\mathbb{R}^n$ and a lifting V^* as follows. First we let V be zero on $f(\Sigma)$ and V^* be zero on $f^{-1}(f(\Sigma))$. Now any continuous extension of V will automatically give a unique lifting V^* on M extending V^* on $f^{-1}(f(\Sigma))$ since outside of Σ the map f is an immersion.

Now extend V over N so that V points outside of ∂N at every point of ∂N. Then V^* will point outside of $f^{-1}(N)$ at the boundary $f^{-1}(\partial N) = \partial f^{-1}(N)$. Now the local index for N_i, which is the component of N containing Σ_i, is $v_i = \chi(\Sigma_i)$. The index of V^* on $f^{-1}(N)$ is equal to $\chi(f^{-1}(N)) = \chi(f^{-1}(f(\Sigma))$.

Now extend V to all of $\mathbb{R}^n$. On the unbounded components of $\mathbb{R}^n - N$ this can be done without introducing a zero. On a bounded component D_j, we may extend V so that there is only one zero at some arbitrary point x_j. We choose $x_j \notin f(\partial M)$ so that V^* is non zero on ∂M.

Now we apply (24) to V^*. First we note that the local index v_j about x_j is equal to $(-1)^n \chi(D_j)$. The $(-1)^n$ comes in since the vector field V is pointing inside at the boundary of $\mathbb{R}^n - \mathring{N}$.

Then (24) gives

$$\Sigma\omega_i(\Sigma_i) + (-1)^n \Sigma\omega_j\chi(D_j) + \chi(M) - \deg \hat{N} = (-1)^n \Sigma n_j \chi(D_j) + \chi\left(f^{-1}(f(\Sigma))\right).$$

Solving for $\chi\left(f^{-1}(f(\Sigma))\right)$ gives the result.

For odd dimension n, equation (29) simplifies to

$$\chi\left(f^{-1}(f(\Sigma))\right) = \Sigma\omega_i\chi(\Sigma_i) + (-1)^n \Sigma(\omega_j - n_j)\chi(D_j). \tag{30}$$

7. Fixed point theory for bodies in space.

Suppose $M^n \subset \mathbb{R}^n$ is a compact manifold. We shall call it a body. Suppose that $f : M^n \to \mathbb{R}^n$ is a continuous map. Then we define a vector field V_f on M^n by

$$V_f(m) = (m - f(m))_m\,. \tag{31}$$

The zeroes of V_f are precisely the fixed points of f. The local index of a zero of V_f is precisely the fixed point index for the fixed point. Now equation (7) and Lemma 3 will combine to give us various formulas analogous to the Lefschetz equation equating the Lefschetz number Λ_f and the sum of the local fixed point indices.

First we consider the example where $f(M) \subset M$, so that f can be regarded as a self map. If f has no fixed points on ∂M, then we obtain equation (3), [1,2,3]

$$\Lambda_f + \Lambda_{-\hat{V}_f,\hat{N}} = \chi(M).$$

Next we suppose that $f : M^n \to \mathbb{R}^n$ is *virtually transverse* on ∂M. That is we suppose the function $\partial M \to \{-1,+1\}$ given by

$$m \mapsto ((m - f(m)) \bullet N(m)) /|(m - f(m)) \bullet N(m)|$$

is well defined and continuous. That is the same as saying that $m - f(m)$ never points both normally inside and outside on the same connected component of ∂M. If the vector $m - f(m)$ were never tangent to ∂M. this is a map *transverse* to ∂M and an example of one virtually transverse to ∂M.

If f is virtually transverse to ∂M, then

$$\operatorname{Ind}(\partial_- V_f) = \Sigma\chi(\partial M_i) \tag{32}$$

where ∂M_i are components of ∂M so that V_f does not point normally outside. For even dimensional M, we see that $\operatorname{Ind}(\partial_- V) = 0$. This gives us the following theorem.

Theorem 13. *If $f : M^n \to \mathbb{R}^n$ is virtually transverse to ∂M, then f has a fixed point if $\chi(M^n) - \Sigma\chi(\partial M_i) \neq 0$.*

Hence
a) If n is even, f has a fixed point if $\chi(M) \neq 0$.
b) If n is odd, f has a fixed point if $\chi(M) \not\equiv 0 (\operatorname{mod} 2)$.

As an example we get Dold's generalization of the Brouwer fixed point theorem, [3]. Namely, if $f : B^n \to \mathbb{R}^n$ is a virtually transverse map of the unit ball, then f has fixed point. Because $\chi(B^n) = 1$.

8. The Euler–Poincaré number and curvature.

Our third example of equation (7) applied to bodies in space relates curvature of ∂M and the Euler–Poincaré numbers of the body M and $M \cap \Pi^k$ where Π^k is a k–dimensional hyperplane. Given Π^k we consider the vector field V given by $V(m) = p(m) - m$ where $p : \mathbb{R}^n \to \Pi^k$ is the orthogonal projection onto Π^k. Consider V restricted to a body M. To use equation (7) we must calculate $\operatorname{Ind} V$ and $\operatorname{Ind} \partial_- V$. First we deal with $\operatorname{Ind} \partial_- V$.

Let $m \in \partial M$. We define the *convexity number* $C(m, \Pi^k)$ as follows. Consider the line normal to ∂M at m. If this line is not normal to Π^k, we let $C(m, \Pi^k) = 0$. If the line is normal to Π^k, we observe that the vector field ∂V must have a zero at m. Then $C(m, \Pi^k)$ is defined to be equal to the index of ∂V at m. If m is not isolated, we define it to be the index of the connected component of the zeroes of ∂V containing m.

With the above definition of $C(m,\Pi^k)$, our next theorem will be correct in every situation. However in the non degenerate smooth situations, we can give a very geometric interpretation of $C(m,\Pi^k)$ which depends on the curvature of ∂M at m. We let ℓ_m denote the ray which begins at m and leaves ∂M in a normal direction. The ray intersects Π^k orthogonally at a point, call it y. Let $\Pi^{\perp}$ denote the $(n-k)$ dimensional hyperplane orthogonal to Π^k and passing through y. Now we consider the $(n-k-1)$ dimensional surface $\Pi^{\perp}\cap\partial M$ in the $(n-k)$ space $\Pi^{\perp}$. This surface contains m and it should have $(n-k-1)$ principal curvatures at m. For each principal curvature there is a center of curvature which lies on the line containing ℓ_m. Suppose that there are r centers of curvature on ℓ_m between y and ∞. In computing r, if two centers of curvature coincide, we count them twice. Then

$$C(m,\Pi^k) = (-1)^r(\mathrm{sign}(m,y)) \tag{33}$$

where $\mathrm{sign}(m,y)$ denotes the sign of the curvature of ∂M at m calculated on the side facing y. For even dimensional ∂M it does not matter which side of ∂M we take to compute the curvature's sign. We are taking the curvature to be the product of the curvatures of the $n-1$ principal curves. The following figure shows these concepts for $\mathbb{R}^2$.

Π^1

ℓ_m $\bullet y$ $\bullet$ $\bullet m$

center of curvature

∂M

In this case $C(m,\Pi^1)=+1$. If we translate Π^1 parallel to itself to the other side of ∂M then we would get $C(m,\Pi^1)=-1$. Now let $\Pi^\circ = y$. In the picture as drawn $C(m,y)=+1$ since $n=0$. As we move y along the line ℓ_m towards the right, the sign $C(m,y)$ changes after we pass the center of curvature and remains negative even after we pass through m since the fact that r changes from 1 to 0 is compensated by the fact that the curvature changes sign as we change sides.

Now we can state our result.

Theorem 14. *Suppose that M^n is a smooth body and Π^k is a k–dimensional hyperplane which is nowhere tangent to ∂M. Then*

$$\chi(M^n) = (-1)^{n-k}\chi\left(M^n\cap\Pi^k\right) + \sum_m C(m,\Pi^k)$$

where the sum is taken over those m for which ℓ_m points initially inward.

Proof. We want to apply (7) to V. The set of zeroes of V is the submanifold $M\cap\Pi^k$. Unfortunately, there are zeroes on ∂M if $\partial M\cap\Pi^k\neq 0$. We will change V slightly to V' such that V' has no zeroes on ∂M and so that $\partial V=\partial V'$. Since

Π^k is nowhere tangent to ∂M, there is an $\varepsilon > 0$ so that $V(m) + \frac{\varepsilon}{2}N(m)$ is never zero when $m \in \partial M$ is of distance less than ε to Π^k. We choose a small collar neighborhood C of ∂M so that $C \cap \Pi^k$ is a collar neighborhood of $\partial M \cap \Pi^k$. We can extend $\frac{\varepsilon}{2}N$ to a vector field W on M so that W is zero on $M - C$ and $V + W$ is never zero on C. Let $V' = V + W$. Then V' has no zeroes on ∂M and $\partial V' = \partial V$.

The zeroes of $\partial V'$ coming from $\partial M \cap \Pi^k$ are now in $\partial_+ V'$, hence $\operatorname{Ind} \partial_- V' = \operatorname{Ind} \partial_- V = \Sigma C(m, \Pi^k)$ where the sum is taken over the inward pointing rays. What is $\operatorname{Ind} V'$? Consider a small compact neighborhood of $M \cap \Pi^k$ of the form $(M \cap \Pi^k) \times I^{n-k}$. Now V' restricted to this neighborhood can be homotopied, without changing the zeroes, to a vector field of othe form $A \times B$, where A is a vector field on $M \cap \Pi^k$ which points outside on $\partial M \cap \Pi^k$, and B is the vector field on I^{n-k} given by $B(m) = -m$. Then $\operatorname{Ind} A = \chi(M \cap \Pi^k)$ and $\operatorname{Ind} B = (-1)^{n-k}$. Since the index of a product of two vector fields is the product of the index, we get

$$\operatorname{Ind} V' = (-1)^{n-k} \chi(M \cap \Pi^k). \tag{34}$$

Combining (34) and (7) gives the theorem.

Note that theorem 14 gives an inductive definition of the Euler–Poincaré number in terms of sections of lower dimensions. As such it extends and illuminates work of [7].

There is another theorem similar in spirit. Let Π^{n-2} be a hyperplane in $\mathbb{R}^n$ and let M^n be a body. Let V be the velocity vector field of a rotation about Π^{n-2}. Thus in $\mathbb{R}^2$, V is the velocity field rotating about a point and in $\mathbb{R}^3$, V is the velocity field rotating about an axis. Where are the zeroes of ∂V? They occur at points $m \in \partial M$ where there is a tangent $(n-1)$–plane which contains Π^{n-2}. We define $d(m, \Pi^{n-2})$ to be the index of ∂V at m. When m is an isolated zero in the non degenerate smooth case, then

$$d(m, \Pi^{n-2}) = \operatorname{sign}(m) \tag{35}$$

where $\operatorname{sign}(m) = +1$ if the curvature is positive when calculated on the side of ∂M to which m is moving under the velocity m. If the curvature is negative, then $\operatorname{sign}(m) = -1$ and if the curvature is zero, then $\operatorname{sign}(m) = 0$.

Now to apply (7) to V we must eliminate the zeroes of V at $\partial M \cap \Pi^{n-2}$. If Π^{n-2} is nowhere tangent to ∂M, we can find a vector field V' as in the proof of theorem 14 so that $\partial V = \partial V'$ and $\operatorname{Ind} \partial_- V' = \operatorname{Ind} \partial_- V'$. This time the homotopy of V' in a compact neighborhood $M \cap \Pi^{n-2} \times I^2$ results in a vector field $A \times B$ so that A on $M \cap \Pi^{n-2}$ has index equal to $\chi(M \cap \Pi^{n-2})$ and B on I^2 has index equal to 1. Thus we get $\operatorname{Ind} V' = \chi(M^n)$ and hence from (7),

Theorem 15. $\chi(M^n) = \chi(M \cap \Pi^{n-2}) + \Sigma d_i(m)$
where Π^{n-2} is nowhere tangent to ∂M and the sum is taken over those points m which are being rotated into M.

As an example of the use of the above theorem we consider the case of a solid torus in three dimensional Euclidean space. Suppose that Π^1 is a line which intersects the torus in two segments. Then the theorem results in the equation $\Sigma d_i(m) = -2$. From this we conclude that there are two points of negative curvature on the boundary of the solid torus whose tangent planes contain the line Π^1. If we reverse the direction of rotation in the theorem we get two other points with the same property. Thus there are four points of negative curvature whose tangent planes contain Π^1.

Now let us play with theorem 14. Suppose we are in a space ship in space and there is a bounded medium M. By shooting out laser beams we can tell which directions are normal to the boundary ∂M and we can determine the convexity numbers by using the reflection at the boundary of the laser beam. Suppose the different amounts of reflection tell us whether or not the normal beam is passing from inside to outside. What can we tell about M?

Answer: We can determine $\chi(M)$ and whether or not we are inside M. The sum of all the convexity numbers gives the $\chi(\partial M)$ and hence $\chi(M) = \frac{1}{2}\chi(\partial M)$. Then using theorem 14 we subtract the sum of the inward pointing convexity numbers from $\chi(M)$ to find $\chi(\Pi^\circ \cap M)$. If it is one, we are inside, if it is zero, we are outside.

We should reflect how lucky we are to live inside an odd dimensional universe. If we were in an even dimension universe we could not use the sum of the convexity numbers to determine $\chi(M)$ and hence we would not know if we were inside or outside M. If we add the capability of measuring the distances to the normal points, then we would know whether we were inside M, since we look at the closest normal point and determine if the beam is exiting or entering M. Then we would know $\chi(\Pi^\circ \cap M)$ and we could determine $\chi(M)$ in even dimensional universes also.

If there were no reflections of beams going from inside to outside and we did not know if we were inside or outside M, then the sums of the convexity numbers we could see would give us either $\chi(M)$ or $\chi(M) + 1$.

Bibliography

1. W. Ambrose, *The index theorem in Riemannian geometry*, Annals of Math. **(2) 73** (1961), 49–86.
2. G. Bredon and A. Kosinski, *Vector fields on* Π*–manifolds*, Annals of Math. **(2) 84** (1966), 85–90.
3. A. Dold, *Algebraic Topology*, Springer, New York, 1972.
4. D.H. Gottlieb, *A de Moivre formula for fixed point theory*, ATAS de 5° Encontro Brasiliero de Topologia **31** (1988), Universidade de Sao Paulo, Sao Carlos S.P., Brasil, 59–67.
5. D.H. Gottlieb, *A de Moivre like formula for fixed point theory*, Proceedings of the fixed point theory seminar at the 1986 International Congress of Mathematics, Robert F. Brown editor, Contemporary Math. Series of the American Math. Soc. **72** (1988), 99–106.
6. D.H. Gottlieb, *On the index of pullback vector fields*, Proceedings of the 2nd Siegen Topology Symposium, August 1987, Ulrich Koschorke editor (to appear in the Springer Lecture Notes).
7. H. Hadwiger, *Eulers charakteristik und kombinatorische geometrie*, J. Reine Angew. Math. (Crelle) **194** (1955), 101–110.
8. A. Haefliger, *Quelques remarques sur les applications différentiables d'une surface dans le plan*, Ann. Inst. Fourier **10** (1960), Grenoble, 47–60.
9. U. Koschorke, *Concordance and bordism of line fields*, Inventiones Math. **24** (1974).
10. J. Milnor, *On the immersion of* n*–manifolds in* $(n+1)$*–space*, Comment. Math. Helv. **30** (1956), 275–284.

11. M. Morse, *Singular points of vector fields under general boundary conditions*, American Journal of Mathematics **51** (1929), 165–178.
12. C. C. Pugh, *A generalized Poincaré index formula*, Topology **7** (1968), 217–226.
13. J. W. Vick, *Homology Theory*, Academic Press, New York.