Functions and the Unity of Mathematics

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Abstract. We give a definition of Mathematics. In the context of this definition we investigate the question: Why does Mathematics appear to have an underlying unity? We suggest in large part it is because of the modern notion of function. We give a brief history of the concept of functions. Then we examine the principle that any general concept easily expressed in terms of functions must appear in many branches of mathematics. We explain the importance of groups and other central concepts by the means of this function principle, and we predict that other new concepts will also come to enjoy the same frequency of application. In particular we discuss the de Moivre–Euler formula from the standpoint of the function principle. We predict another formula will play the the same kind of role. We call it the Law of Vector Fields. We examine the successes of Law of Vector Fields. We show how it gives a new perspective and generalization of the Gauss–Bonnet Theorem. We show how it leads to the classification of vector fields up to homotopy. We show how it leads to an invariant of space–time. We use this invariant to describe a new feature of electo–magnetic fields in terms of the index of vector fields. An examination of classical electromagnetic fields invites us to consider the proposition that the index of the B field can never be nonzero. A consequence of this proposition is that there are no magnetic monopoles; and that there are restrictions on pure B fields beyond being divergence free.

1. The Definition of Mathematics

We take the following definition of Mathematics:

Definition. *Mathematics is the study of well–defined concepts.*

Well–defined concepts are those with no ambiguities. They are clear and distinct. A well-defined concept need not necessarily be defined, however. An important question is: Is the concept of well–defined itself well–defined? After considerable thought, I am inclined to say yes. Hence I take the above definition of Mathematics as mathematical and not merely philosophical.

This definition seems to be novel. I have not found it written in any book or paper. It seems obvious to me, and I believe most mathematicians would subscribe to it. We should treat it seriously, reflect on it, explain it to the world at large.

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Then perhaps we can begin to eliminate the destructive misconceptions that the public at large entertains about mathematics. And we may learn something.

Now well–defined concepts are creations of the human mind. And most of those creations can be quite arbitrary. There is no limit to the well–defined imagination. So if one accepts the definition that Mathematics is the study of the well–defined, then how can Mathematics have an underlying unity? Yet it is a fact that many savants see just such an underlying unity in Mathematics, so the key question to consider is:

Question. *Why does Mathematics appear to have an underlying unity?*

If mathematical unity really exists then it is reasonable to hope that there are a few basic principles which explain the occurrence of those phenomena which persuade us to believe that Mathematics is indeed unified; just as the various phenomena of Physics seem to be explained by a few fundamental laws. If we can discover these principles it would give us great insight into the development of Mathematics and perhaps even insight into Physics.

Now what things produce the appearance of an underlying unity in Mathematics? Mathematics appears to be unified when a concept, such as the Euler characteristic, appears over and over in interesting results; or an idea, such as that of a group, is involved in many different fields and is used in Science to predict or make phenomena precise; or an equation, like De Moivre's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

yields numerous interesting relations among important concepts in several fields in a mechanical way.

Thus the appearance of underlying unity comes from the ubiquity of certain concepts and objects, such as the numbers π and e and concepts such as groups and rings, and invariants such as the Euler characteristic and eigenvalues, which continually appear in striking relationships and in diverse fields of Mathematics and Physics. We use the word *broad* to describe these concepts.

Compare broad concepts with *deep* concepts. The depth of an idea seems to be a function of time. As our understanding of a field increases, deep concepts become elementary concepts, deep theorems are transformed into definitions and so on. But something broad, like the Euler characteristic, remains broad, or becomes broader as time goes on. The relationships a broad concept has with other concepts are forever.

The Function Principle. *Any concept which arises from a simple construction of functions will appear over and over again throughout Mathematics.*

We assert the principle that function is one of the broadest of all mathematical concepts, and any concept or theorem derived in a natural way from that of functions must itself be broad. We will use this principle to assert that the apparent underlying unity of Mathematics at least partly stems from the breadth of the concept of function. We will show how the breadth of category and functor and equivalence and e and π and de Moivre's formula and groups and rings and Euler Characteristic all follow from this principle. We will subject this principle to the rigorous test of a scientific theory: It must predict new broad concepts. We make such predictions and report on evidence that the predictions are correct.

2. The History of Functions

On page 13 of Howard Eves mathematical history [E] there is a picture of the Ishango bone. Carved on the 8,000 year old bone are notches. It is obvious that this bone was used for counting. Each notch was assigned to some object; a day, an eclipse, a cow. This bone is the physical representation of some function. There is evidence of this sort from 50,0000 years ago. Humans were probably employing functions long before there were words for numbers.

Over the centuries function–like concepts entered into mathematics and its applications. The chord tables of Ptolemy, and his maps of the world, are two examples from ancient times. Oresme around 1360 studied "latitudes of forms" which was a graphical representation of functions [Bo]. The Renaissance artist–mathematicians and mapmaker–mathematicians introduced projections. Galileo groped at the concept in describing the outcomes of his experiments and his physical laws. Leibniz first used the word function in 1694 to describe quantities that varied along a curve, [Kl]. Controversies arose between Euler and d'Alembert and others about which functions could count as solutions to the wave equations or be represented by a trigonometric series, [Ka]. Slowly piecewise continuous, and then discontinuous functions were accepted. Complex functions and various symmetry transformations were accepted by 1870. Then Cantor introduced the generality of sets and one to one correspondences. Soon one could think about functions on arbitrary sets. As for continuous functions, Bolzano essentially gave the correct definition in 1817 for functions between the real numbers , but his work was ignored. The ϵ–δ definition is due to Weierstass in 1859. The topological definition was given by Hausdorff in 1914, [Kl].

Thus the concept of a function as a mapping $f\colon X \to Y$ from a source set X to a target set Y did not develop until the Twentieth Century. The modern concept of a function did not even begin to emerge until the middle ages. The beginnings of Physics should have given a great impetus to the notion of function, since the measurements of the initial conditions of an experiment and the final results gives implicitly a function from the initial states of an experiment to the final outcomes; but historians say that the early physicists and mathematicians never thought this way. Soon thereafter calculus was invented. For many years afterwards functions were thought to be always given by some algebraic expression. Slowly the concept of a function of a mapping grew. Cantor's set theory gave the notion a good impulse but the modern notion was adopted only in the Twentieth Century. See [ML] for a good account of these ideas.

3. The Unity of Mathematics

The careful definition of function is necessary so that the definition of the composition of two functions can be defined. Thus $f \circ g$ is only defined when the target of g is the source of f. This composition is associative: $(f \circ g) \circ h = f \circ (g \circ h)$ and f composed with the identity of either the source or the target is f again. We call a set of functions a *category* if it is closed under compositions and contains the identity functions of all the sources and targets.

Category was first defined by S. Eilenberg and S. MacLane and was employed by Eilenberg and N. Steenrod in the 1940's to give homology theory its functorial character. Category theory became a subject in its own right, it's practitioners joyfully noting that almost every branch of Mathematics could be organized as a category. The usual definition of category is merely an abstraction of functions closed under composition. The functions are abstracted into things called morphisms and composition becomes an operation on sets of morphisms satisfying exactly the same properties that functions and composition satisfy. Most mathematicians think of categories as very abstract things and are surprised to find they come from such a homely source as functions closed under composition.

A *functor* is a function whose source and domain are categories and which preserves composition. That is, if F is the functor, then $F(f \circ g) = F(f) \circ F(g)$. This definition also is abstracted and one says category and functor in the same breath.

Question. *What statements can be made about a function f which would make sense in every possible category?*

There are basically only four statements since the only functions known to exist in every category are the identity functions. We can say that f is an identity, or that f is a *retraction* by which we mean that there is a function g so that $f \circ g$ is an identity, or that f is a *cross–section* by which we mean that there is a function h so that $h \circ f$ is an identity, or finally that f is an *equivalence* by which we mean that f is both a retraction and a cross–section. In the case of equivalence the function h must equal the function g and it is called the *inverse* of f and it is unique.

Retraction and cross–section induce a partial ordering of the sources and targets of a category, hereafter called the *objects* of the category. Equivalences induce an equivalence relation on the objects and give us the means of making precise the notion that two mathematical structures are the same.

Now consider the self equivalences of some object X in a category of functions. Since X is both the source and the target, composition is always defined for any pair of functions, as are inverses. Thus we have a *group*. The definition of a group in general is just an abstraction, where the functions become undefined elements and composition is the undefined operation which satisfies the group laws of associativity and existence of identity and inverse, these laws being the relations that equivalences satisfy. The notion of functor restricted to a group becomes that of *homomorphism*. The equivalences in the category of groups and homomorphisms are called *isomorphisms*.

The concept of groups arose in the solution of polynomial equations, with the first ideas due to Lagrange in the late eighteenth century, continuing through Abel to Galois. Felix Klein proposed that Geometry should be viewed as arising from groups of symmetries in 1872 in his Erlanger Programm. Poincare proposed that the equations of Physics should be invariant under the correct symmetry groups around 1900. Since then groups have played an increasingly important role in Mathematics and in Physics. The increasing appearance of this broad concept must have fed the feeling of the underlying unity of Mathematics. Now we see how naturally it follows from the Function Principle.

If we consider a set of functions S from a fixed object X into a group G, we can induce a group structure on S by defining the multiplication of two functions f and g to be $f * g$ where $f * g(a) = f(a) \cdot g(a)$. Here a runs through all the elements in X and "$\cdot$" is the group multiplication in G. This multiplication can be easily shown to satisfy the laws of group multiplication. The same idea applied to maps into the Real Numbers or the Complex Numbers gives rise to addition and multiplication on functions. These satisfy properties which are abstracted into the concepts of abelian rings. If we consider the set of self homomorphisms of abelian groups and use composition and addition of functions, we get an important example of a non–commutative ring. The natural functors for rings should be ring homomorphisms. In the case of a ring of functions into the Real or Complex numbers we note that a ring homomorphism h fixes the constant maps. If we consider all functions which fix the constants and preserve the addition, we get a category of functions from rings to rings; that is, these functions are closed under composition. We call these functions *linear transformations*. They contain the ring homomorphisms as a subset. Study the equivalences of this category. We obtain the concepts of *vector spaces* and linear transformations after the usual abstraction.

Now we consider a category of homomorphisms of abelian groups. We ask the same question which gave us equivalence and groups,

Question. *What statements can be made about a homomorphism f which would make sense in every possible category of abelian groups?*

Now between every possible abelian group there is the trivial homomorphism $0: A \to B$ which carries all of A onto the identity of B. Also we have for every integer N the homomorphism from A to itself which adds every element to itself N times, that is multiplication by N.

Thus for any homomorphism $h: A \to B$ there are three statements we can make which would always make sense. First $N \circ h$ is the trivial homomorphism 0, second that there is a homomorphism $\tau: B \to A$ so that $h \circ \tau$ is multiplication by N, and third that $\tau \circ h$ is multiplication by N. So we can give to any homomorphism three non–negative integers: The *exponent*, the *cross–section degree*, and the *retraction degree*. The *exponent* is the smallest positive integer such that $N \circ h$ is the trivial homomorphism 0. If there is no such N then the exponent is zero. Similarly the *cross–section degree* is the smallest positive N such that there is a τ, called a *cross–section transfer*, so that $h \circ \tau$ is multiplication by N. Finally the *retraction degree* is the smallest positive N such that there is a τ, called a *retraction transfer*, so that $\tau \circ h$ is multiplication by N.

In accordance with the Function Principle, we predict that these three numbers will be seen to be broad concepts. Their breadth should be less than the breadth of equivalence, retraction and cross–section because the concepts are valid only for categories of abelian groups and homomorphisms. But exponent, cross–section degree and retraction degree can be pulled back to other categories via any functor from that category to the category of abelian groups. So these integers potentially can play a role in many interesting categories. In fact for the category of topological spaces and continuous maps we can say that any continuous map $f: X \to Y$ has exponent N or cross–section degree N or retraction degree N if the induced homomorphism $f_*: H_*(X) \to H_*(Y)$ on integral homology has exponent N or cross–section degree N or retraction degree N respectively.

As evidence of the breadth of these concepts we point out that for integral homology, cross–section transfers already play an important role for fibre bundles. There are natural transfers associated with many of the important classical invariants such as the Euler characteristic and the index of fixed points and the index of vector fields,[BG],[G1] and the Lefschetz number and coincidence number and most recently the intersection number, [GO]. And a predicted surprise relationship occurs in the case of cross–section degree for a map between two spaces. In the case that the two spaces are closed oriented manifolds of the same dimension, the cross–section degree is precisely the absolute value of the classical Brouwer degree. The retraction degree also is the Brouwer degree for closed manifolds if we use cohomology as our functor instead of homology, [G1].

A common activity in Mathematics is solving equations. There is a natural way to frame an equation in terms of functions. In an equation we have an expression on the left set equal to an expression on the right and we want to find the value of the variables for which the two expressions equal. We can think of the expressions as being two function f and g from X to Y and we want to find the elements x of X such that $f(x) = g(x)$. The solutions are called *coincidences.* Coincidence makes sense in any category and so we would expect the elements of any existence or uniqueness theorem about coincidences to be very broad indeed. But we do not predict the existence of such a theorem. Nevertheless in Topology there is such a theorem. It is restricted essentially to maps between closed oriented manifolds of the same dimension. It asserts that locally defined coincidence indices add up to a globally defined coincidence number which is given by the action of f and g on the homology of X. In fact this coincidence number is the alternating sum of traces of the composition of the umkehr map $f_!$, which is defined using Poincare Duality, and g_*, the homomorphism induced by g. We predict, at least in Topology and Geometry, more frequent appearances of both the coincidence number and also the local coincidence index and they should relate with other concepts.

If we consider self maps of objects, a special coincidence is the fixed point $f(x) = x$. From the point of view of equations in some algebraic setting, the coincidence problem can be converted into a fixed point problem, so we do not lose any generality in those settings by considering fixed points. In any event the fixed point problem makes sense for any category. Now the relevant theorem in Topology is the Lefschetz fixed point theorem. In contrast to the coincidence theorem, the Lefschetz theorem holds essentially for the wider class of compact spaces. Similar to the coincidence theorem, the Lefschetz theorem has locally defined fixed point indices which add up to a globally defined Lefschetz number. This Lefschetz number is the alternating sum of traces of f_*, the homomorphism induced by f on homology. This magnificent theorem is easier to apply than the coincidence theorem and so the Lefschetz number and fixed point index are met more frequently in interesting situations than the coincidence number and coincidence indices.

In other fields fixed points lead to very broad concepts and theorems. A linear operator gives rise to a map on the one dimensional subspaces. The fixed subspaces are generated by *eigenvectors.* Eigenvectors and their associated eigenvalues play an important role in Mathematics and Physics and are to be found in the most surprising places.

Consider the category of C^∞ functions on the Real Line. The derivative is a function from this category to itself taking any function f into f'. The derivative practically defines the subject of calculus. The fixed points of the derivative are multiples of e^x. Thus we would predict that the number e appears very frequently in calculus and any field where calculus can be employed. Likewise consider the set of analytic functions of the Complex Numbers. Again we have the derivative and its fixed point are the multiples of e^z. Now it is possible to relate the function e^z

defined on a complex plane with real valued functions by

$$e^{(a+ib)} = e^{a}(\cos(b) + i\sin(b)).$$

We call this equation de Moivre's formula. This formula contains an unbelievable amount of information. Just as our concept of space–time separation is supposed to break down near a black hole in Physics, so does our definition–theorem view of Mathematics break down when considering this formula. Is it a theorem or a definition? Is it defined by sin and cos or does it define those two functions?

Up to now the function principle predicted only that some concepts and objects will appear frequently in undisclosed relationships with important concepts throughout Mathematics. However the de Moivre equation gives us methods for discovering the precise forms of some of the relationships it predicts. For example, the natural question "When does e^z restrict to real valued functions?" leads to the "discovery" of π. From this we might predict that π will appear throughout calculus type Mathematics, but not with the frequency of e. Using the formula in a mechanical way we can take complex roots, prove trigonometric identities, etc.

There is yet another fixed point question to consider: What are the fixed points of the identity map? This question not only makes sense in every category; it is solved in every category! The invariants arising from this question should be even broader than those from the fixed point question. But they are very uninteresting. However, if we consider the fixed point question for functions which are equivalent to the identity under some suitable equivalence relation in a suitable category we may find very broad interesting things. A suitable situation involves the fixed points of maps homotopic to the identity in the topological category. For essentially compact spaces the *Euler characteristic* (also called the Euler–Poincare number) is an invariant of a space whose nonvanishing results in the existence of a fixed point. This Euler characteristic is the most remarkable of all mathematical invariants. It can be defined in terms simple enough to be understood by a school boy, and yet it appears in many of the star theorems of Topology and Geometry. A restriction of the concept of the Lefschetz number, its occurrence far exceeds that of its "parent" concept. possibly first encountered by Descartes,[St], then used by Euler to study regular polyhedra, the Euler characteristic slowly proved its importance. Bonnet showed in the 1840's that the total curvature of a closed surface equaled a constant times the Euler characteristic. Poincare gave it its topological invariance by showing it was the alternating sum of Betti numbers. In the 1920's Lefschetz showed that it determined the existence of fixed points of maps homotopic to the identity, thus explaining, according to the Function Principle, its remarkable history up to then and predicting the astounding frequency of its subsequent appearances in Mathematics.

The Euler characteristic is equal to the sum of the local fixed point indices of the map homotopic to the identity. We would predict frequent appearances of the local index. Now on a smooth manifold we consider vector fields and regard them as representing infinitesimally close maps to the identity. Then the local fixed point index is the local index of the vector field.

These considerations lead us to the prediction that a certain equation due to Marston Morse, [M], will play a very active role in Mathematics, and by extension Physics. This equation, which we call the Law of Vector Fields, was discovered in

1929 and has not played a role at all commensurate with our prediction up until now.

We describe the Law of Vector fields. Let M be a compact manifold with boundary. Let V be a vector field on M with no zeros on the boundary. Then consider the open set of the boundary of M where V is pointing inward. Let $\partial_- V$ denote the vector field defined on this open set on the boundary which is given by projecting V tangent to the boundary. The Euler characteristic of M is denoted by $\chi(M)$, and $\mathrm{Ind}(V)$ denotes the index of the vector field. Then the Law of Vector Fields is

$$\mathrm{Ind}(V) + \mathrm{Ind}(\partial_- V) = \chi(M)$$

We propose two methods of applying the law of vector fields to get new results and we report on their successes. These successes and the close bond between Physics and Mathematics encourage us to predict that the Law of Vector Fields and its attendant concepts must play a vital role in Physics.

4. The Law of Vector Fields

Prediction. *Just as de Moivre's formula gives us mechanical methods which yields precise relationships among broad concepts, we predict that the Law of Vector Fields will give mechanical methods which will yield precise relationships among broad concepts in Mathematics and Physics.*

Method one.

1. Choose an interesting vector field V and manifold M.
2. Adjust the vector field if need be to eliminate zeros on the boundary.
3. Identify the global and local Ind V.
4. Identify the global and local index $\mathrm{Ind}(\partial_- V)$.
5. Substitute 3 and 4 into the Law of Vector Fields.

We predict that this method will succeed because the Law of Vector Fields is morally the definition of index, so all features of the index must be derivable from that single equation. We measure success in the following descending order: 1. An important famous theorem generalized; 2. A new proof of an important famous theorem; 3. A new, interesting result. We put new proofs before new results because it may not be apparent at this time that the new result will famous or important.

In category 1 we already have the extrinsic Gauss-Bonnet theorem of differential Geometry [G3], the Brouwer fixed point theorem of Topology [G3], and Hadwiger's formulas of Integral Geometry, [G3], [Had], [Sa]. In category 2 we have the Jordan separation theorem, The Borsuk-Ulam theorem, the Poincare-Hopf index theorem of Topology; Rouche's theorem and the Gauss-Lucas in complex variables; the fundamental theorem of algebra and the intermediate value theorem of elementary Mathematics; and the not so famous Gottlieb's theorem of group homology, [G2]. Of course we have more results in category 3, but it is not so easy to describe them with a few words. One snappy new result is the following: *Consider any straight line and smooth surface of genus greater than 1 in three dimensional Euclidean space. Then the line must be contained in a plane which is tangent to the surface,* ([G3], theorem 15).

We will discuss the Gauss-Bonnet theorem since that yields results in all three categories as well as having the longest history of all the results mentioned. One of the most well-known theorems from ancient times is the theorem that the sum of the angles of a triangle equals π. If we consider exterior angles we see they add up to 2π . In fact, that is true for any polygon. This in turn is the limiting case of the theorem which states that the tangent vector on a simply closed curve in a plane sweeps out an angle of 2π. The normal vector also sweeps out an angle of 2π. This can be rephrased as *The Gauss map has degree equal to one for a simple closed curve.*

Gauss showed for a triangle whose sides are geodesics on a surface M in three-space that the sum of the angles equals $\pi + \int_M KdM$, where K is the Gaussian curvature of the surface. Bonnet pieced these triangles together to prove that for a closed surface M the total curvature $\int_M KdM$ equals $2\pi\chi(M)$. Hopf proved that $\int_M KdM$, where M is a closed hypersurface in odd dimensional Euclidean space and K is the product of the principal curvatures must equal the degree of the Gauss map $\hat{N} : M^{2n} \to S^{2n}$ times the volume of the unit sphere. Then he proved $2\deg(\hat{N}) = \chi(M^{2n})$. (Morris Hirsch in [Hi] gives credit to Kronicker and Van Dyck for Hopf's result in dimension 2.) According to [Br], Lefschetz then showed that $deg\hat{N} = \chi(M)$ where $\hat{N}$ is the Gauss map for the boundary ∂M for any co–dimension zero manifold M in Euclidean space as a consequence of his fixed point theorem. The degree of the Hopf map is what Hopf called the Curvatura Integrala, and is proportional to the integral of the product of the principal curvatures.

At this point something amazing happened. The Gauss–Bonnet theorem was hijacked. In order to prove an intrinsic version of the Hopf–Gauss–Bonnet theorem, the theorem was hijacked. The odd dimensional case was excluded. Thus the "generalized Gauss–Bonnet Theorem" no longer contains the case of the plane triangle. Allendorfer and Fenchel both proved the "generalized Gauss–Bonnet Theorem" in 1940 and Chern in 1944 gave an intrinsic proof to the intrinsic ""generalized Gauss–Bonnet Theorem".

From our viewpoint today we can see what happened if we replace the question of finding a statement and proof the Gauss–Bonnet Theorem for an abstract $2n$–dimensional manifold M by alternative questions. Allendorfer and Fenchel solved the question of finding an equation for $deg\hat{N}$ in terms of the Riemannian curvature of M embedded in Euclidean space, and Chern answered the question of finding a formula for the Euler characteristic of M in terms of Riemannian curvature of M. For a history of the Gauss-Bonnet theorem see [Gr], pp. 89-72 or [Sp], p. 385.

Now the questions above lead to a wonderful theorem, but it should not be called the Gauss–Bonnet Theorem. From the topological point of view, a more reasonable question is: *Find a formula for $deg\hat{N}$ for a manifold which is immersed in one higher dimensional Euclidean space.* Method one produces the answer.

The Gauss–Bonnet Theorem. *Let $f : M \to R^n$ be a smooth map from a compact Riemannian manifold of dimension n to n-dimensional Euclidean space so that f near the boundary ∂M is an immersion. And let ∂M be orientable. The index of the gradient of $x \circ f : M \to R$, where x is the projection of R^n onto the x-axis, is equal to the difference between the Euler Characteristic and the degree of the Gauss map. Thus*

$$\mathrm{Ind}(\mathrm{grad}(x \circ f)) = \chi(M) - \deg\hat{N}.$$

The proof runs as follows following method one. The interesting vector field is $V = grad(x \circ f)$. The zeros of $\partial_- V$ are the coincidence points of the Gauss maps $\hat{N}$ and $-\hat{V}$, where the Gauss map of $-V$ is defined by making $-V$ of unit length and parallel translating it from ∂M to the unit sphere. Now coincidence theory tells us that the total coincidence number is the difference between the two degrees. Since V has no zeros, its Gauss map has zero degree. Thus $\operatorname{Ind} \partial_- V = \deg \hat{N}$. Plug this into the Law of Vector Fields.

This equation leads to an immediate proof of the Gauss-Bonnet Theorem, since for odd dimensional M and any vector field W, the index satisfies $\operatorname{Ind}(-W) = -\operatorname{Ind}(W)$. Thus the left side of the equation reverses sign while the right side of the equation remains the same. Thus $\chi(M)$ equals the degree of the Gauss-map, which is the total curvature over the volume of the standard $n-1$ sphere. Now $2\chi(M) = \chi(\partial M)$, so we get Hopf's version of the Gauss-Bonnet theorem.

Note as a by-product we also get $\operatorname{Ind}(\operatorname{grad}(x \circ f)) = 0$ which is a new result thus falling into category 3. Another consequence of the generalized Gauss-Bonnet theorem follows when we assume the map f is an immersion. In this case the gradient of $x \circ f$ has no zeros, so its index is zero so the right hand side in zero and so again $\chi(M) = \deg \hat{N}$. This is Haefliger's theorem [Hae], a category 2 result. Please note in addition that the Law of Vector Fields applied to odd dimensional closed manifolds, combined with the category 2 result $\operatorname{Ind}(-W) = -\operatorname{Ind}(W)$, implies that the Euler characteristic of such manifolds is zero, (category 2). So the Gauss-Bonnet theorem and this result have the same proof in some strong sense. Finally we mention that any closed oriented manifold which can be immersed in a co-dimension one Euclidean space is a boundary, thus the conditions of the above theorem can always be obtained.

Just as the Gauss-Bonnet theorem followed from considering gradient vector fields, the Brouwer fixed point theorem is generalized by considering the following vector field. Suppose M is an n-dimensional body in R^n and suppose that $f : M \to R^n$ is a continuous map. Then let the vector field V_f on M be defined by drawing a vector from m to the point $f(m)$ in R^n. If the map f satisfies the transversal property, that is the line between any m on the boundary of M and $f(m)$ is not tangent to ∂M at m, then f has a fixed point if $\chi(M)$ is odd (category 1). This last sentence is an enormous generalization of the Brouwer fixed point theorem, yet it remains a small example of what can be proved from applying the Law of Vector Fields to V_f. In fact the Law of Vector Fields applied to V_f is the proper generalization of the Brouwer fixed point theorem.

Method two. *Make precise the statement that the Law defines the index of vector fields.*

In this method we learn from the Law. The Law teaches us that there is a generalization of homotopy which is very useful. This generalization, which we call *otopy*, not only allows the vector field to change under time, but also its domain of definition changes under time. An otopy is what $\partial_- V$ undergoes when V is undergoing a homotopy. A proper otopy is an otopy which has a compact set of zeros.

More precisely: An *otopy* is a vector field V defined on the closure of an open set $T \subset M \times I$ so that $V(m, t)$ is tangent to the slice $M \times t$. The otopy is *proper* if the set of defects D of V is compact and contained in T. The restriction of V to

$M \times 0$ and $M \times 1$ are said to be properly otopic vector fields. Proper otopy is an equivalence relation.

Index Classifies Otopy Classes of Vector Fields. *Let M be a connected manifold. The proper otopy classes of proper vector fields on M are in one to one correspondence via the index to the integers. If M is a compact manifold with a connected boundary, then a vector field V is properly homotopic to W if and only if* $\mathrm{Ind}\ V = \mathrm{Ind}\ W$. *In general, V is properly homotopic to W if and only if* $\mathrm{Ind}(\partial_- V) = \mathrm{Ind}(\partial_- W)$ *on each connected component of the boundary ∂M.* [GS]

Over one hundred years ago Poincare introduced the index of a vector field. He knew it was invariant under proper homotopy. Now we discover that it is classified by proper otopy, a concept given to us by the Law of Vector Fields. Another victory for the Function Principle. But there is even more, we can classify the proper homotopy classes of a vector field.

We repeat. The proper otopy classes of vector fields on a connected manifold are in one to one correspondence with the integers via the map which takes a vector field to its index. This leads to the fact that homotopy classes of vector fields on a manifold with a connected boundary where no zeros appear on the boundary are in one to one correspondence with the integers. This is not true if the boundary is disconnected.

The generality of otopy and the Law of Vector Fields suggests we can extend the setting for the definition of index. We find that we do not need to assume that vector fields are continuous. We can define the index for vector fields which have discontinuities and which are not defined everywhere. We need only assume that the set of "defects" is compact and never appears on the boundary or frontier of the sets for which the vector fields are defined. We then can define an index for any compact connected component of defects (subject only to the mild condition that the component is open in the subspace of defects). Thus under an otopy, it is as if the defects change shape with time and collide with other defects, and all the while each defect has an integer associated with it. This integer is preserved under collisions. That is, the sum of the indices going into a collision equals the sum of the indices coming out of a collision, provided no component "radiates out to infinity", i.e. loses its compactness, [GS].

This picture is very suggestive of the way charged particles are supposed to interact. Using the Law of Vector Fields as a guide we have defined an index which satisfies a conservation law under collisions. The main ideas behind the construction involve dimension, continuity, and the concept of pointing inside. We suggest that those ideas might lie behind all the conservation laws of collisions in Physics.

5. Vector Bundles and Lorentzian Manifolds

The generality of the concept of otopy suggests we can consider even more general settings for the Index of vector fields. We can study vector fields along the fibre on fibre bundles. An otopy generalizes to a vector field along the fibre V restricted to an open set. For a proper V only certain values of the index of V restricted to a fibre are possible. For example the Hopf fibrations of spheres admit only the index zero. These restrictions arise, as the Function Principle foretells, because of a transfer associated with the index of V restricted to a fibre.

We make this precise. A useful generalization of proper otopy is that of a proper vector field along the fibres of a fibre bundle

$$M \to E \to B$$

where M is a smooth manifold and V is a vector field along the fibres and proper means that the defects of V are compact over any compact subset of B. Then V restricted to any fibre has the same index (if B is connected). In this view the fibre bundle is like a collections of possible proper otopies. We view an otopy as a vector field changing under time. A proper vector field along the fibres restricts the possible proper otopies. This occurs because of the following transfer theorem, [BG].

Transfer Theorem.
Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a smooth fibre bundle with F a compact manifold and B a closed manifold. Let V be a proper vector field on E with vectors tangent to the fibres. Then there is an S–map $\tau : B^+ \to E^+$ so that in ordinary homology $p_* \circ \tau_*$ (cohomology $\tau^* \circ p^*$) is multiplication by the Index of V restricted to a fibre, $\mathrm{Ind}(V|F)$.

A second general setting for otopy and the index are Lorentzian manifolds. Here a space–like vector field V on an open set is a generalized otopy; and this otopy gives an equivalence relation between the vector fields which are projections of V on space–like slices, [GS].

Space–time invariance of Index. *Suppose V is a space–like vector field in a space–time S. Suppose M and N are two time–like slices of S which can be smoothly deformed into each other. Suppose D, the set of defects of V, is compact in the region of S where the deformation takes place. Then the index of V projected onto M is equal to the index of V projected onto N.*

proof. We can set up a proper otopy between the two projected vector fields given the hypotheses of the theorem. First we consider the mapping $F : M \times I \longrightarrow S$. On each slice F is a smooth embedding. On $M \times \{0\}$ F restricts to the identity and on $M \times \{1\}$, F restricts to an embedding onto N. The vector field k on $M \times I$ tangent to I will map onto time–like vectors in S, so that the pullback V' of V onto $M \times I$ will be space–like on $M \times I$. Then we project V' to a space–like vector field W by subtracting the k component from V'. So W is an otopy and on the slice $M \times \{0\}$ it is exactly the projection of V on M. On the other hand, the restriction of W to $M \times \{1\}$ need not correspond to the projection of V on N (under the identification of N with $M \times \{1\}$) because k need not map onto the normal vectors n of N. However $tn + (1-t)k$ is a homotopy between them, so W on N is homotopic to the projection of V on N. No new zeros will be created by this process since V is space–like, so the otopy is proper. Thus the projection of V on M is properly homotopic to the projection of V on N. $\square$

6. Physics

We have the following picture immerging out of the previous sections. A vector field has a set of connected components of defects. Now under a homotopy these components move around and collide with one another. There is a conservation law, which says that the sum of the indices of the components going into a collision is equal to the sum of the indices of the components at the collision is equal to the sum of the indices after the collision, if during the homotopy all the components remain compact and there are only a finite number of them, [GS]. Thus the index remains conserved unless some component "radiates out to infinity." This mathematical result is very reminiscent of the action of charged particles with the index of the defect playing the role of the charged particle.

The fact that charge–like conservation follows from a simple topological construct, which depends only on continuity and dimension and pointing inside, suggests that the topological concept of index may have physical content.

There is another compelling reason to consider the index as a physical quantity. It is an invariant of General Relativity.

We can study space-like vector fields on a Lorentzian space-time. The otopy generalizes to a space-like vector field restricted to an open set. For a proper space-like vector field V the index of V projected onto to a space-like slice (by means of a normal time-like vector field to the slice) is constant. Thus the index is an invariant of General Relativity. For example, if we consider the Newtonian gravitational field for our solar system in a ball which extends far out beyond the matter of the solar system, and if we consider this as a space-like slice of space time, then the index of the gravitational field, which is -1, remains -1 no matter which space-like slice is substituted.

Space-like vector fields arise in several situations. If u is a unit time-like vector field, then the covariant derivative of u is a space-like vector field. Thus the index of the covariant derivative is an invariant. If u represents a continuous family of observers, then the covariant derivative a represents the field of "gravitational acceleration", and the index is an invariant for all space-like slices.

We may study 2–forms using the index. Let F be a 2–form on space-time S. Let $\hat{F}$ denote the associated linear transformation on the tangent space of S. Let u denote a time-like unit vector field. Then there is a vector field associated to u which is a space-like vector field given by $\overrightarrow{E} = \hat{F}u$. Now consider the 2–form $*F$. Here the $*$ denotes the Hodge dual which depends on the choice of orientation made on S. Now another vector field relative to u is given by $-\overrightarrow{B} = (\hat{*F})u$. Note that $\overrightarrow{B}$ reverses direction if the orientation is changed.

Now both $\overrightarrow{E}$ and $\overrightarrow{B}$ are space-like vector fields orthogonal to u since F is skew symmetric with respect to the Lorentzian metric of S. Thus for any 2–form F and a choice u of a time-like vector field, we have two integers Ind $\overrightarrow{E}$ and Ind $\overrightarrow{B}$. These integers are independent of which space-like slice they are calculated on. Note however that the sign of Ind $\overrightarrow{B}$ depends on the choice of orientation of S .

Now suppose that v is another unit time-like vector field on S. We assume that u and v are both future pointing. Then the homotopy $tu + (1-t)v$ passes through future pointing time-like vector fields which are never zero if u and v are never zero. Thus $\overrightarrow{E_u}$ and $\overrightarrow{B_u}$ is homotopic to $\overrightarrow{E_k}$ and $\overrightarrow{B_k}$. If this is a proper homotopy, then the two indices for F and u agree with those for F and k. If the homotopy is proper for all k, then the pair of integers (Ind $\overrightarrow{E}$, Ind $\overrightarrow{B}$) depends only on F.

We can tell when the pair (Ind $\overrightarrow{E}$, Ind $\overrightarrow{B}$) is well–defined. If the set of points of S so that $\hat{F}$ has a non-trivial kernel is compact, then any homotopy between different future pointing unit vector fields leads to a proper homotopy, because no new zeros can be created on the points where $\hat{F}$ is nonsingular. Now $\hat{F}$ is nonsingular if and only if $\overrightarrow{E_u} \bullet \overrightarrow{B_u} \neq 0$ for any choice of u. If $E = B \neq 0$, then the kernel of $\hat{F}$ is nowhere time-like, so no new zeros are created at those points. Let D_E be the closure of the set of points where $\overrightarrow{E_u} \bullet \overrightarrow{B_u} = 0$ and $B_u > E_u$ and let D_B be the closure of the set of points where $\overrightarrow{E_u} \bullet \overrightarrow{B_u} = 0$ and $B_u < E_u$. If D_E and D_B are compact we say that F is *proper.* We have established the following theorem.

Index for proper 2–forms. *Let F be a two–form such that D_E and D_B are compact. Then the pair of integers* $(\mathrm{Ind}\,\overrightarrow{E}, \mathrm{Ind}\,\overrightarrow{B})$ *is well–defined.*

If F is an electro-magnetic 2–form, we can calculate $(\mathrm{Ind}\,\overrightarrow{E}, \mathrm{Ind}\,\overrightarrow{B})$ where now $\overrightarrow{E}$ and $\overrightarrow{B}$ are the electric and magnetic vector fields respectively. So these two integers are properties of any suitable F and it is reasonable to use them as part of the description of electro-magnetism.

It is not difficult to get the index of a vector field using the Law of Vector Fields. The Coulomb electric vector field E of an electron or a proton has index -1 or 1 respectively. G. Samarayanake, in her thesis [Sam], has a computer program which estimates the index of a zero using the Law of Vector Fields. Using this program she can search for zeros of a static coulomb electric field generated with a finite number of electrons and protons whose index is not -1, 0, or 1. Placing protons at the vertices of the Platonic solids; Tetrahedron, Octahedron, Cube, Icosahedron, and Dodecahedron, she estimates the index of the central zero to be –3, –5, 5, –11, and 11 respectively.

Since these indices are easy to find and since they describe some aspect of classical electro-magnetic fields, they should be used to describe physical phenomena.

Here is a true and meaningful statement using index. *E fields with nonzero indices are more common than B fields with nonzero indices.* In fact, it is tempting to conjecture: *There are no B fields with finite nonzero indices.*

This conjecture implies that not every solution to Maxwell's equations is physical, since it is possible to find a divergence free vector field which has nonzero finite index. Thus certain pure B fields should not exist according to our conjecture. Note the theory that electro-magnetic fields have point like sources also argues against the existence of pure B fields. The conjecture also implies that magnetic monopoles do not exist.

Now if two magnetic monopoles of index 1 are placed near each other then there is a zero of the B field of index -1. So we should look for B zeros with nonzero index. If this cannot be done, it argues against the existence of magnetic monopoles. If it can be done, maybe magnetic monopoles will be present, since if magnetic monopoles were present, these zeros would be present. Also the existence or non-existence of these B zeros with nonzero index would add evidence about whether pure B fields can exist.

Mathematically the conjecture can be restated as follows: *Every proper B field is properly otopic to every other one and this is true regardless of the choice of orientation of space-time used to represent either of the B fields. This means there is only one otopy class for B fields and this is the same whichever choice of orientation.* If the conjecture were false, there would be an infinite number of otopy classes of B fields and they would be sensitive to choice of orientation.

Let us see how easy it is to get non-zero indices of E fields. The proton has index 1. The electron has index -1. Put a surface of genus g made out of a conductor inside a large imaginary ball. Charge the conductor with a positive charge. Then $\operatorname{Ind} \overrightarrow{E}$ inside the sphere is equal to $2-2g$. For a negatively charged surface the index equals $2g-2$.

On the other hand, none of the B fields in [FLS] has nonzero index. If we try the trick with surfaces of genus g made out of super conductors, we find that the B field is tangent to the surface. This is opposite to the above case where the E field is normal to the conducting surface. Since the B field is tangent to the surface, there is always a zero on the surface (unless $g=1$). Thus $\operatorname{Ind} B = \infty$ except when the surface is a torus, in which case $\operatorname{Ind} B = 0$.

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